Degenerate optomechanical parametric oscillators: Cooling in the vicinity of a critical point

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Degenerate optomechanical parametric oscillators are optical resonators in which a mechanical degree of freedom is coupled to a cavity mode that is nonlinearly amplified via parametric down-conversion of an external pumping laser. Below a critical pumping power the down-converted field is purely quantum mechanical, making the theoretical description of such systems very challenging. Here we introduce a theoretical approach that is capable of describing this regime, even at the critical point itself. We find that the down-converted field can induce significant mechanical cooling and identify the process responsible of this as a cooling-by-heating mechanism. Moreover, we show that, contrary to naive expectations and semiclassical predictions, cooling is not optimal at the critical point, where the photon number is largest. Our approach opens the possibility of analyzing further hybrid dissipative quantum systems in the vicinity of critical points.

I. INTRODUCTION

Degenerate optical parametric oscillators (DOPOs) consist of a driven optical cavity containing a crystal with second-order optical nonlinearity [1–4]. Down-conversion in the crystal can generate a field at half the frequency of the driving laser and classical electrodynamics predicts that this field will start oscillating inside the cavity only if the external laser power exceeds some threshold value, where the nonlinear gain can compensate for the cavity losses. A fully quantum-mechanical theory, on the other hand, reveals that even below threshold the down-converted field is not a vacuum, but a squeezed field whose quantum correlations increase as the threshold is approached.

Recent developments in the fabrication of crystalline whispering-gallery-mode (cWGM) resonators [5–17] have opened the way to study of the intracavity interplay between down-conversion and optomechanics [18], a setup that we refer to as the *degenerate optomechanical parametric oscillator* (DOMPO). So far, it has been shown that the presence of down-conversion in an optomechanical cavity can help to enhance mechanical cooling [19], normal mode splitting [20], and sensitivity in position measurements [21] or can even bring optomechanics close to the strong-coupling regime with additional bath engineering [22]. In all these works, however, the nonlinear crystal is operated as a parametric amplifier, providing a nonlinear gain to some external field that is injected in the cavity at the down-converted frequency (*stimulated* down-conversion). In contrast, the description of the interaction between the field generated via *spontaneous* down-conversion and the mechanical mode is much more challenging, since (below threshold) the former is purely quantum mechanical [23], so that the optomechanical coupling cannot be linearized and does not admit a simple Gaussian description.

In this work we provide a theory for the DOMPO which can be trusted all the way to threshold and is obtained by combining traditional adiabatic elimination techniques with our recently developed self-consistent Mori projector (c-MoP) theory [24,25]. To this end we first introduce the master equation which models the DOMPO and perform an adiabatic elimination of the optical modes by neglecting the mechanical backaction. The mechanical state is found to stay approximately thermal for parameters compatible with current cWGM resonators, with an effective temperature dependent on the steady-state value and two-time correlation function of the down-converted photon number, which we derive in two ways. First, we treat the pump mode as a classical field (semiclassical approach), allowing us to obtain simple analytical expressions and provide a physical explanation for the regions of significant cooling, showing that the system provides a realistic implementation of the cooling-by-heating mechanism [26] below threshold. Second, we use c-MoP theory on the optical dynamics to find reliable results at threshold and justify the absence of the mechanical backaction onto the optics. Remarkably, this accurate approach allows us to prove that the semiclassical predictions break down when working very close to threshold, where cooling is shown to disappear. These results might have strong implications not only for future analysis, but also for previous results which make use of semiclassical approaches while working very close to threshold [22]. In the final section we apply c-MoP theory to the full optomechanical problem and identify the region of the parameter space where the mechanical backaction on the optics is negligible (which contains the parameters of our interest).

II. THE DEGENERATE OPTOMECHANICAL PARAMETRIC OSCILLATOR

The system we consider is schematically represented in Fig. 1. A crystal with second-order optical nonlinearity is shared by two cavities with relevant resonances at frequencies ω_p (pump) and $\omega_s \approx \omega_p/2$ (signal). The pump cavity is driven by a resonant laser, so that photons in the signal cavity

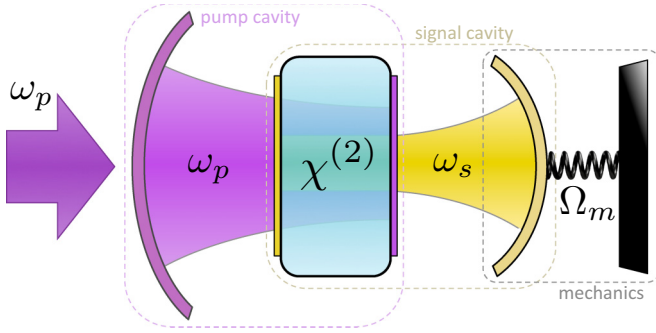*peter.degenfeld-schonburg@ph.tum.de
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FIG. 1. Sketch of the degenerate optomechanical parametric oscillator. We take crystalline whispering-gallery-mode resonators [5–17] as a reference for our choice of parameters, but electromechanical implementations are also possible using superconducting circuits in the degenerate parametric oscillation configuration [27] coupled to drum-shaped oscillating capacitors [28–30] (see also [31]).

can be generated via *spontaneous down-conversion* [1,4]. In addition, one of the mirrors of the signal cavity can oscillate and is therefore optomechanically coupled to the down-converted field via radiation pressure [18]. Let us define the annihilation operators $\{a_j\}_{j=p,s,m}$ for the pump (p), signal (s), and mechanical (m) modes. Including losses of the optical modes at rate γ_0 (assumed to be the same for pump and signal without loss of generality), as well as the irreversible energy exchange of the mechanical mode with its thermal environment at rate γ_m (with which, in the absence of light, it is in thermal equilibrium with $\bar{n}_{\text{th}}$ phonons), the master equation governing the evolution of the DOMPO's state ρ can be written as

$$\dot{\rho} = -i[H,\rho] + \gamma_0 \mathcal{D}_{a_s}[\rho] + \gamma_0 \mathcal{D}_{a_p}[\rho]$$
$$+ \gamma_m(\bar{n}_{\text{th}} + 1)\mathcal{D}_{a_m}[\rho] + \gamma_m \bar{n}_{\text{th}} \mathcal{D}_{a_m^\dagger}[\rho]. \quad (1)$$

We have defined the Lindblad superoperators $\mathcal{D}_J[\cdot] = 2J(\cdot)J^\dagger - J^\dagger J(\cdot) - (\cdot)J^\dagger J$ and the Hamiltonian

$$H = H_{\text{opt}} + \Omega_m a_m^\dagger a_m - \Omega_m \eta_{\text{OM}} a_s^\dagger a_s (a_m^\dagger + a_m), \quad (2)$$

where we normalize the optomechanical coupling η_{OM} to the frequency of the mechanical oscillation Ω_m. The optical Hamiltonian can be written in a picture rotating at the laser frequency as

$$H_{\text{opt}} = \Delta_s a_s^\dagger a_s + i\epsilon_p(a_p^\dagger - a_p) + i\frac{\chi}{2}\left(a_p a_s^{\dagger 2} - a_p^\dagger a_s^2\right), \quad (3)$$

where $\Delta_s = \omega_s - \omega_p/2$ is the detuning of the signal mode (which we take positive in this work), $\chi/2$ is the down-conversion rate, and ϵ_p is proportional to the square root of the injected laser power.

In the classical limit, the steady-state phase diagram of the DOMPO features a variety of phases [23]. Here we focus on the regime where the state of the signal field is fully quantum, i.e., where the trivial solution $\langle a_s \rangle = 0$ is the only stable one, henceforth referred to as the *monostable phase*, which requires two conditions. First, defining the *injection parameter* $\sigma = \epsilon_p \chi / \gamma_0^2$ and the normalized detuning $\Delta = \Delta_s / \gamma_0$, the trivial solution becomes unstable in favor of a nontrivial one, $\langle a_s \rangle \neq 0$, for $\sigma > \sqrt{1 + \Delta^2}$ [23]. Hence, we write $\sigma = \sqrt{1 + \Delta^2}\, x$ and focus on the $x \in [0,1]$ region. Second, the condition $4\Omega\Delta\eta_{\text{OM}}^2 / \eta_{\text{DC}}^2 < 1$ guarantees that the nontrivial

solution does not enter the $x \in [0,1]$ region [23]. In this expression we have introduced the dimensionless down-conversion coupling $\eta_{\text{DC}} = \chi / \gamma_0$ as well as the sideband-resolution parameter $\Omega = \Omega_m / \gamma_0$.

We emphasize that the vanishing signal-field amplitude excludes the possibility of using a linearization approach similar to those applied in [19–21] and [32–36]. In the following, we provide a theory that works in the entire $x \in [0,1]$ region and use it to predict the action of the down-converted field on the mechanical state.

III. EFFECTIVE MECHANICAL DYNAMICS

Despite the complexity of the problem, we remarkably find, with the help of c-MoP theory, that for typical system parameters the optical modes do not receive considerable backaction from the mechanics. This property, which we justify in Sec. VI, allows us to simplify the problem significantly via an adiabatic elimination of the optical modes [32,37–40], leading to an effective master equation for the reduced mechanical state $\rho_m(t)$. As we show in Appendix A, the mechanical steady state can then be approximated by a thermal state (displaced by $\langle a_m \rangle \approx \eta_{\text{OM}} \bar{N}_s$) characterized by its phonon number,

$$\bar{n}_m = \frac{\bar{n}_{\text{th}} + \Gamma_+}{1 + (\Gamma_- - \Gamma_+)} \overset{\Gamma \gg 1}{\approx} \frac{\bar{n}_{\text{th}}}{\Gamma} + \bar{n}_{\text{FL}}, \quad (4)$$

where $\Gamma = \Gamma_- - \Gamma_+$ is the cooling efficiency and $\bar{n}_{\text{FL}} = \Gamma_+ / \Gamma$ the fundamental limit for the phonon number. All the information on the optical modes is contained in the heating and cooling rates $\Gamma_\pm = C \operatorname{Re}\{\gamma_0 \int_0^\infty d\tau \exp(\mp i\Omega_m \tau)\, s(\tau)\}$ through the optical correlation function

$$s(\tau) = \operatorname{tr}\{a_s^\dagger a_s\, e^{\mathcal{L}_{\text{opt}}\tau} a_s^\dagger a_s\, \bar{\rho}_{\text{opt}}\} - \bar{N}_s^2, \quad (5)$$

where $\bar{N}_s = \operatorname{tr}\{a_s^\dagger a_s \bar{\rho}_{\text{opt}}\}$ is the signal photon number and $C = \Omega Q \eta_{\text{OM}}^2$ the bare cooperativity, with $Q = \Omega_m / \gamma_m$ the mechanical quality factor. Here, $\bar{\rho}_{\text{opt}}$ is the steady state of the optical Liouvillian,

$$\mathcal{L}_{\text{opt}}[\cdot] = -i[H_{\text{opt}}, \cdot] + \sum_{j=s,p} \gamma_0 \mathcal{D}_{a_j}[\cdot]; \quad (6)$$

that is, $\mathcal{L}_{\text{opt}}[\bar{\rho}_{\text{opt}}] = 0$.

In the following we study the behavior of the steady-state phonon number as we approach the DOMPO's threshold. From Eq. (4) it is clear that optimal cooling is then found by simultaneously maximizing Γ and minimizing $\bar{n}_{\text{FL}}$.

The nonlinear nature of the parametric down-conversion process in Eq. (6) and a potential backaction of the mechanical mode preclude an exact treatment of the optical correlation function in Eq. (5). To get simple analytic expressions that enable physical insight, we first apply standard linearization to the optical problem, which we denote the *semiclassical* approach and has been the method of choice in previous works [19–22]. Next, applying c-MoP theory [25] we show the failure of the semiclassical approach close to the critical point and find more accurate expressions at criticality, which also allow us to justify the adiabatic elimination of the optical modes.

IV. SEMICLASSICAL APPROACH

Below threshold, the linearization of the DOPO is accomplished by treating the pump mode as a classical stationary source, that is, by performing the replacement $a_p \to \epsilon_p/\gamma_0$ [2–4]. Within this approximation, the optical problem is governed by a Gaussian single-mode Liouvillian,

$$\gamma_0^{-1}\mathcal{L}_{\mathrm{opt}}[\cdot] = -i\big[\Delta a_s^\dagger a_s + \sigma\big(a_s^{\dagger 2} - a_s^2\big)/2, \cdot\big] + \mathcal{D}_{a_s}[\cdot], \quad (7)$$

from which any correlation function can easily be found, allowing us to obtain analytical expressions for the relevant quantities in Eq. (4), as we show in Appendix B. For the fundamental limit, we find

$$\bar{n}_{\mathrm{FL}} = [4 + (\Omega - 2\Delta)^2]/8\Omega\Delta, \quad (8)$$

while the cooling efficiency can be written as

$$\Gamma = Q\eta_{\mathrm{OM}}^2 \bar{N}_s(x)\Delta f(\Omega,\delta_{\mathrm{eff}}), \quad (9)$$

where we have defined the function

$$f(\Omega,\delta_{\mathrm{eff}}) = \frac{8\Omega^2\big[\Omega^2 + 4\big(5 + \delta_{\mathrm{eff}}^2\big)\big]}{(4 + \Omega^2)\big[\Omega^4 + 16\big(1 + \delta_{\mathrm{eff}}^2\big)^2 + 8\Omega^2\big(1 - \delta_{\mathrm{eff}}^2\big)\big]}$$

and a parameter, $\delta_{\mathrm{eff}} = \sqrt{\Delta^2 - \sigma^2}$, that is shown later to play the important role of an effective optical detuning. The photon number $\bar{N}_s(x) = x^2/(2 - 2x^2)$ is fully due to quantum fluctuations and increases hyperbolically until the threshold $x = 1$, where it diverges in this semiclassical approach.

In Fig. 2 we show the steady-state phonon number as a function of the two control parameters, detuning Δ and distance to threshold x, fixing the rest of the parameters to typical values of cWGM resonators [9,10]. There are two regions where significant cooling effects appear. One is in the vicinity of the threshold point and can be traced back to the vast increase in the photon number $\bar{N}_s$, which makes $\Gamma \gg 1$ for virtually any value of the remaining parameters. However, as we show below with the c-MoP approach, this close to threshold the semiclassical approach breaks down, hence rendering this prediction incorrect.

The other region, which turns out to be of major significance when aiming for optimal cooling, corresponds to $\delta_{\mathrm{eff}} \approx \Omega/2$ (see the solid black line in Fig. 2). The c-MoP approach confirms this prediction in the next section. Moreover, it can be understood in physical terms by moving to a new picture defined by the squeezing operator $S(r) = \exp[-ir(a_s^{\dagger 2} + a_s^2)/2]$, with $\tanh 2r = \sigma/\Delta$ (note that this transformation requires $\Delta > \sigma$, which corresponds in Fig. 2 to the region above the dashed black line). This transformation diagonalizes the Hamiltonian in the optical Liouvillian, (7), so that, defining the parameters $\bar{N}_{\mathrm{eff}} = (\Delta/\delta_{\mathrm{eff}} - 1)/2$ and $M = \sigma/2\delta_{\mathrm{eff}}$, the transformed state $\tilde{\rho} = S^\dagger(r)\rho S(r)$ evolves according to

$$\gamma_0^{-1}\partial_t\tilde{\rho}$$
$$= [-i\delta_{\mathrm{eff}}a_s^\dagger a_s, \tilde{\rho}] + (\bar{N}_{\mathrm{eff}} + 1)\mathcal{D}_{a_s}[\tilde{\rho}] + \bar{N}_{\mathrm{eff}}\mathcal{D}_{a_s^\dagger}[\tilde{\rho}]$$
$$- [i\Omega a_m^\dagger a_m, \tilde{\rho}] + (\Omega/Q)\{(\bar{n}_{\mathrm{th}} + 1)\mathcal{D}_{a_m}[\tilde{\rho}] + \bar{n}_{\mathrm{th}}\mathcal{D}_{a_m^\dagger}[\tilde{\rho}]\}$$
$$+ \big[\Omega\eta_{\mathrm{OM}}M\big(a_s^2 a_m^\dagger - a_s^{\dagger 2}a_m\big), \tilde{\rho}\big], \quad (10)$$

within the rotating-wave approximation, valid under the conditions $4\delta_{\mathrm{eff}}^2 \gg \sigma$ and $\eta_{\mathrm{OM}}\Delta \ll \delta_{\mathrm{eff}}$ (see Appendix C).
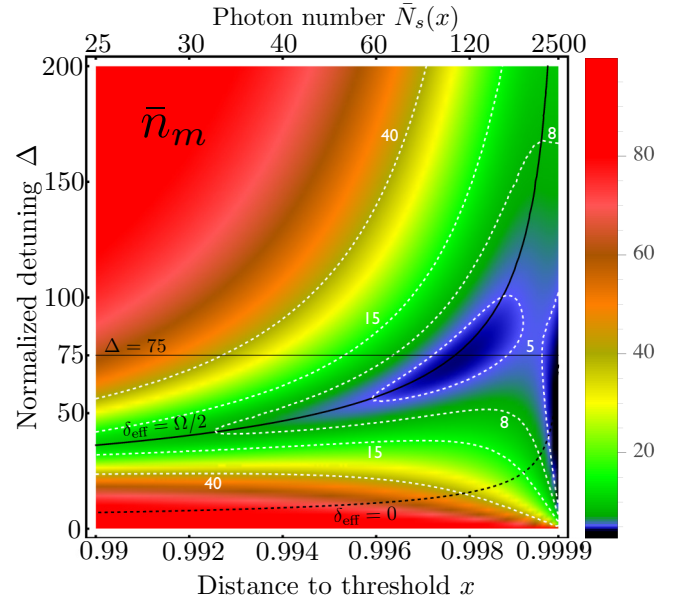

FIG. 2. Steady-state phonon number as a function of Δ and x, as obtained from a semiclassical description of the DOPO and a thermal phonon number $\bar{n}_{\mathrm{th}} = 100$ (achievable either with cryogenics [41] or via a standard sideband precooling stage). We choose typical parameters for cWGM resonators [10]: $\Omega = 10$, $\eta_{\mathrm{OM}} = 10^{-4}$, and $Q = 10^6$. Note that typical optical decay rates are in the megahertz range, although only normalized quantities are relevant for our results. The signal steady-state photon numbers $\bar{N}_s(x)$ corresponding to the ticked x values are shown on the upper axis, showing that cooling is effective even with just ~100 photons. On the other hand, c-MoP theory has allowed us to prove that the cooling region closer to the threshold disappears once this more accurate approach is used (see Fig. 3).

Therefore, in this picture the signal field is turned into a bosonic mode with oscillation frequency δ_{eff} and thermal occupation $\bar{N}_{\mathrm{eff}}$. The optomechanical coupling is dressed by the squeezing parameter M, similarly to the dressing by the intracavity-field amplitude in standard sideband cooling [32,40]. However, differently from that case, the interaction exchanges phonons with pairs of photons (rather than single photons), thus explaining why $\delta_{\mathrm{eff}} = \Omega/2$ is the resonance condition for cooling. Under this condition, assuming that $2\bar{N}_{\mathrm{eff}} \gg 1$ and $\Omega^2 \gg 4$, we, furthermore, find $\Gamma \approx 2CM^2\bar{N}_{\mathrm{eff}}$. The cooling efficiency Γ thus receives an additional contribution $2\bar{N}_{\mathrm{eff}}$ from the effective thermal photon number, which is a direct consequence of the nonlinear nature of the effective optomechanical coupling in (10), which cannot be found in standard sideband cooling, as we discuss in Appendix C. This represents a natural example of the so-called *cooling-by-heating* effect [26], where heating up the optical field can contribute to making optomechanical cooling more efficient. However, as is well known from standard sideband cooling [42], thermal photons also contribute to the fundamental limit, which indeed can be approximated by $\bar{n}_{\mathrm{FL}} \approx \bar{N}_{\mathrm{eff}}/2$ in our scenario. When the term $\bar{n}_{\mathrm{th}}/\Gamma$ dominates over $\bar{n}_{\mathrm{FL}}$ in Eq. (4) the thermal optical background $\bar{N}_{\mathrm{eff}}$ can be interpreted as "good noise," while as soon as the fundamental limit is reached it becomes "bad noise" and heats up the
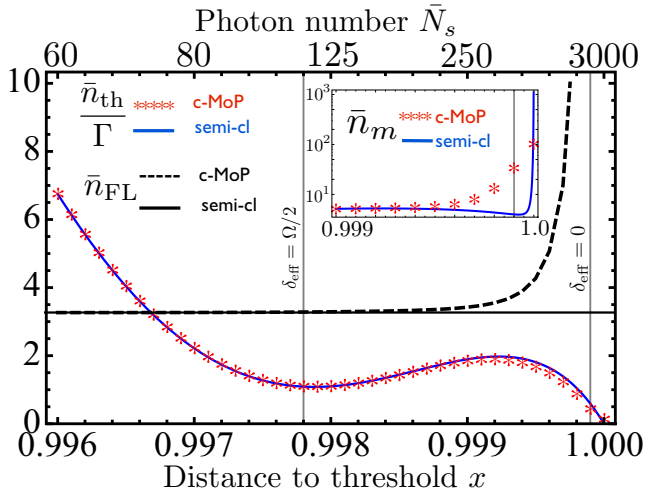
FIG. 3. The terms $\bar{n}_{\text{th}}/\Gamma$ and $\bar{n}_{\text{FL}}$ contributing to the steady-state phonon number, (4), as a function of the distance-to-threshold parameter x. We fix the detuning to $\Delta = 75$, corresponding to the thin solid horizontal line in Fig. 2, taking $\eta_{\text{DC}} = 0.01$ [9] and the rest of the parameters as in Fig. 2. Inset: Closeup of the steady-state phonon number $\bar{n}_m$ for $x \in [0.999, 1]$, but without applying the rotating-wave and Markov approximations [see Eq. (A5)]. Note that c-MoP gives finite results equivalent to those found within the rotating-wave approximation, while the semiclassical predictions diverge at the threshold, which can be taken as further evidence that the theory breaks down there.

mechanical motion (see the thick black curve in Fig. 2). We discuss this phenomenon in more detail in Appendix C.

It is interesting to examine the limits that this bad noise imposes on cooling. For the parameters in Fig. 3, compatible with current cWGM resonators, it is found that the cooling efficiency and the fundamental limit can be simultaneously optimized to about $\Gamma \approx 85$ and $\bar{n}_{\text{FL}} \approx 3$ as a function of the experimentally tunable parameters Δ and σ, precluding ground-state cooling. However, the foreseeable increase in the mechanical quality factor and the optomechanical coupling by one order of magnitude in next-generation cWGM resonators [10] will improve these numbers to $\Gamma \approx 400$ and $\bar{n}_{\text{FL}} \approx 0.15$.

V. c-MoP APPROACH

The semiclassical approach has allowed us to gain analytical and physical insight into the problem. It is, however, well known that this approximation fails close to the critical point, although there is no systematic way of checking where exactly within the semiclassical formalism itself. Hence, to determine exactly where it breaks down and to find more accurate results for those parameters, we make use of the recently developed c-MoP technique [24,25], which allows us to find reduced equations for the constituent parts of a composite system, even in situations where there is significant backaction among its parts and no time-scale separation between their dynamics.

For parameters compatible with cWGM resonators, the theory is already regularized by using c-MoP only in the optical problem (DOPO), which provides a more accurate description of the optical correlation function, (5), and photon

number that enter the effective mechanical dynamics. The application of c-MoP to the DOPO has been detailed in [25], but we review its most relevant steps for completeness in Appendix D. Specifically, we use a combination of c-MoP and a Gaussian-state approximation, which provides an efficient and accurate tool capable of regularizing the divergencies and unphysical predictions of the semiclassical approach. In particular, we show in Appendix D that at threshold the decay rate of the optical correlator scales as $\gamma_{\text{opt}} \propto \gamma_0 \eta_{\text{DC}}(1 + \Delta)$, and the photon number as $\bar{N}_s \propto (1 + \Delta)/\eta_{\text{DC}}$, in contrast to semiclassical results in which the former goes to 0 while the latter diverges.

We show a very representative case for the phonon number $\bar{n}_m$ as a function of the distance to threshold x in Fig. 3. The method shows perfect agreement with the semiclassical predictions sufficiently below threshold, in particular, verifying the cooling-by-heating effect presented above. Most importantly, we find that the absolute minimum phonon number is indeed reached when the resonance condition $\delta_{\text{eff}} = \Omega/2$ is met. On the other hand, close to threshold we find a significant correction with the semiclassical predictions for the fundamental limit $\bar{n}_{\text{FL}}$. In particular, while this is independent of the distance to threshold x in the semiclassical picture, c-MoP shows that it actually increases very rapidly as the critical point is approached, and hence no cooling is found no matter how much the efficiency Γ is increased. This is consistent with the fact that when $\Delta < \sigma$ (as happens at threshold), δ_{eff} becomes imaginary and there is no resonance for the optomechanical interaction.

VI. ABSENCE OF MECHANICAL BACKACTION ON THE OPTICS

The adiabatic elimination of the optical fields which we have used throughout the work relies on the time-scale separation between the optical and the mechanical degrees of freedom. In particular, this approach neglects mechanical backaction on the optics, which is a good approximation as long as the rate of any mechanical perturbation is much lower than the intrinsic relaxation rate of the optics γ_{opt}. Far from the critical point the optical relaxation rate is γ_0, which usually dominates over any other rate in the system. However, as the critical point is approached the DOPO dynamics exhibits a critical slowing-down, and its relaxation rate becomes lower and lower. Hence, in our work, which considers parameters close to threshold, it is very important to check that the desired time-scale separation is present.

An intuitive argument supporting such a time-scale separation follows from relating the mechanical backaction rate to the optical frequency shift induced by the optomechanical interaction, $\gamma_{\text{back}} = \eta_{\text{OM}}\Omega_m\langle x_m \rangle = 2\eta_{\text{OM}}^2\Omega_m\bar{N}_s$, where we have used (A8). Hence, using the scaling of $\bar{N}_s$ and γ_{opt} obtained in the previous section at threshold, the condition $\gamma_{\text{back}} \ll \gamma_{\text{opt}}$ becomes $2\Omega(\eta_{\text{OM}}/\eta_{\text{DC}})^2 \ll 1$, which is very well satisfied for the parameters that we work with. Moreover, note that this condition is automatically satisfied when working within the monostability condition $4\Delta\Omega(\eta_{\text{OM}}/\eta_{\text{DC}})^2 < 1$ as long as $\Delta \gg 1/2$.

We can set more rigorous bounds to the region where mechanical backaction is negligible by using c-MoP theory

[24,25], since, in contrast to adiabatic elimination methods, it does not rely on the concept of time-scale separation or absence of backaction effects. Hence, we apply this theory to the DOMPO system by using the time-dependent c-MoPs $\mathcal{P}_t^{\text{opt}} = \rho_{\text{opt}}(t) \otimes \text{tr}_{\text{opt}}\{\cdot\}$ and $\mathcal{P}_t^m = \text{tr}_m\{\cdot\} \otimes \rho_m(t)$, that is, using a bipartition "optics $\otimes$ mechanics" for the system. This approach allows us to identify the terms contributing to the mechanical backaction and find upper bounds to their scaling. Following the procedure introduced in previous works [24,25], the c-MoP equations for the reduced optical and mechanical states in the asymptotic $t \to \infty$ or steady-state limit are easily found to read

$$
\frac{d\bar{\rho}_m}{dt} = 0 = \mathcal{L}_m \bar{\rho}_m + i\Omega_m \eta_{\text{OM}} \langle a_s^\dagger a_s \rangle [x_m, \bar{\rho}_m]
$$
$$
- \Omega_m^2 \eta_{\text{OM}}^2 \left[x_m, \int_0^\infty d\tau \, e^{\mathcal{L}_m \tau} (\delta x_m \bar{\rho}_m s(\tau) - \text{H.c.}) \right],
$$
$$(11a)$$

$$
\frac{d\bar{\rho}_{\text{opt}}}{dt} = 0 = \mathcal{L}_{\text{opt}} \bar{\rho}_{\text{opt}} + i\Omega_m \eta_{\text{OM}} \langle x_m \rangle [a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]
$$
$$
- \Omega_m^2 \eta_{\text{OM}}^2 \left[a_s^\dagger a_s, \int_0^\infty d\tau \, e^{\mathcal{L}_{\text{opt}} \tau} (\delta n_s \bar{\rho}_{\text{opt}} \, s_m(\tau) - \text{H.c.}) \right],
$$
$$(11b)$$

where, for any operator A, we have introduced the usual definitions $\langle A \rangle = \text{tr}\{A(\bar{\rho}_s \otimes \bar{\rho}_p)\}$ and $\delta A = A - \langle A \rangle$, while $\delta n_s = a_s^\dagger a_s - \langle a_s^\dagger a_s \rangle$. $s(\tau)$ is the usual optical correlation function, (5), and we have defined the mechanical correlation function

$$
s_m(\tau) = \text{tr}_m\{x_m e^{\mathcal{L}_m \tau} \delta x_m \bar{\rho}_m\} = e^{(-i\Omega_m - \gamma_m)\tau}
$$
$$
+ \left[e^{(-i\Omega_m - \gamma_m)\tau} \left(\langle \delta a_m^\dagger \delta a_m \rangle + \langle \delta a_m^2 \rangle \right) + \text{H.c.} \right], \quad (12)
$$

where the final expression is easily found by following a procedure similar to the one reported in Appendix B for the semiclassical optical correlation function, since $\mathcal{L}_m$ is quadratic.

The last two terms on the right-hand side of Eq. (11b) account for the mechanical backaction on the optics, and we proceed now to bound their effect. The second-to-last term describes precisely the optical detuning $\Omega_m \eta_{\text{OM}} \langle x_m \rangle$ induced by the optomechanical interaction that we discuss at the beginning of the section, concluding that it is negligible within our parameter regime. Then we focus on the last term in Eq. (11b), the Born term. For this purpose we derive the steady-state equation for the moment $\langle a_s^{\dagger 2} \rangle$ (note that the equation of motion of $\langle a_s^\dagger a_s \rangle$ receives no explicit mechanical backaction even within c-MoP theory), which reads

$$
0 = \text{tr}_{\text{opt}}\{a_s^{\dagger 2} \mathcal{L}_{\text{opt}} \bar{\rho}_{\text{opt}}\} - 2i\Omega_m \eta_{\text{OM}} \langle x_m \rangle \langle a_s^{\dagger 2} \rangle
$$
$$
+ 2\Omega_m^2 \eta_{\text{OM}}^2 \int_0^\infty d\tau \, \text{Re}\{s_m(\tau)\} \, \text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} [a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]\}
$$
$$
+ 2i\Omega_m^2 \eta_{\text{OM}}^2 \int_0^\infty d\tau \, \text{Im}\{s_m(\tau)\} \, \text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} \{\delta n_s, \bar{\rho}_{\text{opt}}\}\},
$$
$$(13)$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. First, note that the correlation functions $\text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} [a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]\}$ and

$\text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} \{\delta n_s, \bar{\rho}_{\text{opt}}\}\}$, which have a structure similar to that of the optical correlation function $s(\tau)$, decay to 0 at a rate γ_{opt}. Next, we derive upper bounds for the last two terms in Eq. (13). For the second-to-last term we find

$$
\left| 2\Omega_m^2 \eta_{\text{OM}}^2 \int_0^\infty d\tau \, \text{Re}\{s_m(\tau)\} \, \text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} [a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]\} \right|
$$
$$
\leqslant \left| 2\Omega_m^2 \eta_{\text{OM}}^2 \frac{\text{Re}\{s_m(0)\} \, \text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} 0} [a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]\}}{\gamma_m + \gamma_{\text{opt}}} \right|
$$
$$
\leqslant \underbrace{\frac{4\Omega_m^2 \eta_{\text{OM}}^2 \bar{n}_m}{\gamma_m + \gamma_{\text{opt}}}}_{\gamma_{\text{back}}'} |\langle a_s^{\dagger 2} \rangle|, \quad (14)
$$

where in the last step we have used $s_m(0) = 1 + \langle \delta a_m^\dagger \delta a_m \rangle + \langle \delta a_m^2 \rangle \approx \langle \delta a_m^\dagger \delta a_m \rangle \equiv \bar{n}_m$ (note that we expect the mechanical state to stay approximately thermal, and hence $\langle \delta a_m^2 \rangle \approx 0$) and $\text{tr}_{\text{opt}}\{a_s^{\dagger 2}[a_s^\dagger a_s, \bar{\rho}_{\text{opt}}]\} = \langle [a_s^{\dagger 2}, a_s^\dagger a_s] \rangle = 2\langle a_s^{\dagger 2} \rangle$. Similarly, for the last term in Eq. (13) we find

$$
\left| 2i\Omega_m^2 \eta_{\text{OM}}^2 \int_0^\infty d\tau \, \text{Im}\{s_m(\tau)\} \, \text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} \tau} \{\delta n_s, \bar{\rho}_{\text{opt}}\}\} \right|
$$
$$
\leqslant \left| 2\Omega_m^2 \eta_{\text{OM}}^2 \frac{\text{tr}_{\text{opt}}\{a_s^{\dagger 2} e^{\mathcal{L}_{\text{opt}} 0} \{\delta n_s, \bar{\rho}_{\text{opt}}\}\}}{\gamma_m + \gamma_{\text{opt}}} \right|
$$
$$
\leqslant \frac{2\Omega_m^2 \eta_{\text{OM}}^2}{\gamma_m + \gamma_{\text{opt}}} \left| \text{tr}_{\text{opt}}\{a_s^{\dagger 2} \{\delta n_s, \bar{\rho}_{\text{opt}}\}\} \right|
$$
$$
\approx \underbrace{\frac{4\Omega_m^2 \eta_{\text{OM}}^2 (2\bar{N}_s + 1)}{\gamma_m + \gamma_{\text{opt}}}}_{\gamma_{\text{back}}''} |\langle a_s^{\dagger 2} \rangle|, \quad (15)
$$

where for the last expression we have used $\text{tr}_{\text{opt}}\{a_s^{\dagger 2}\{\delta n_s, \bar{\rho}_{\text{opt}}\}\} = 2(\langle a_s^{\dagger 3} a_s \rangle - \langle a_s^{\dagger 2} \rangle \langle a_s^\dagger a_s \rangle + \langle a_s^{\dagger 2} \rangle) \approx 2(2\bar{N}_s + 1)\langle a_s^{\dagger 2} \rangle$, within the Gaussian-state approximation, (B6), that is, $\langle a_s^{\dagger 3} a_s \rangle \approx 3\langle a_s^{\dagger 2} \rangle \langle a_s^\dagger a_s \rangle$, noting that $\langle a_s \rangle = 0$ below threshold.

A sufficient condition for mechanical backaction to be negligible is then $\gamma_{\text{back}}', \gamma_{\text{back}}'' \ll \gamma_{\text{opt}}$. We proceed to check whether this is the case in our work. Note first that $\gamma_m \ll \gamma_{\text{opt}}$ even at threshold, since $\gamma_{\text{opt}}/\gamma_m \sim \gamma_0 \eta_{\text{DC}}(1 + \Delta)/\gamma_m \gg 1$ for the parameters we are interested in. Using the scalings $\gamma_{\text{opt}} \propto \gamma_0 \eta_{\text{DC}}(1 + \Delta)$ and $\bar{N}_s \propto (1 + \Delta)/\eta_{\text{DC}}$ at threshold (where these bounds are the tightest), we can then write the conditions under which backaction is negligible as

$$
\frac{\gamma_{\text{back}}'}{\gamma_{\text{opt}}} \sim \frac{\Omega^2 \bar{n}_m \eta_{\text{OM}}^2}{\eta_{\text{DC}}^2 (1 + \Delta)^2} \ll 1, \quad (16a)
$$

$$
\frac{\gamma_{\text{back}}''}{\gamma_{\text{opt}}} \sim \frac{\Omega^2 \eta_{\text{OM}}^2}{\eta_{\text{DC}}^3 (1 + \Delta)} \ll 1. \quad (16b)
$$

For the parameter set in Fig. 2 these lead to the conditions $\bar{n}_m \ll 100(1 + \Delta)^2$ and $1 + \Delta \gg 1$, respectively. For the large values of Δ that we use throughout most of the work, these conditions are very well satisfied. For small Δ they seem to be too tight, but we need to stress here that we have been extremely conservative when estimating the Born terms, (14) and (15),

meaning that, in practice, backaction should be negligible even in a much broader region of the parameter space.

Overall, c-MoP theory has allowed us to quantify the mechanical backaction on the optics in a rigorous manner. We have obtained very conservative bounds that the system parameters must satisfy in order for such backaction to be negligible, showing that this is indeed the case for the parameters used in our work, which are compatible with an implementation based on cWGM resonators. It is, however, foreseeable that such devices, as well as their electromechanical counterparts, will be able to study regions where backaction is significant, in which case the c-MoP approach presented in this section will be very useful.

VII. CONCLUSIONS

By exploiting adiabatic elimination techniques, semiclassical methods, and c-MoP theory, we have provided a theoretical analysis of the DOMPO which works even at the critical point. We have focused on the region where the optical field is fully quantum, showing that such a quantum-correlated field with no coherent component can induce significant mechanical cooling through a cooling-by-heating mechanism. c-MoP techniques have allowed us to check the validity of the optical adiabatic elimination as well as the semiclassical approximation, whose predictions have indeed been shown to break down at threshold, showing the potential of c-MoP to treat dissipative quantum-optical problems in the vicinity of critical points.

ACKNOWLEDGMENTS

APPENDIX A: ELIMINATION OF THE OPTICAL MODES

Here we present a derivation of the effective mechanical master equation leading to the phonon number of Eq. (4) in the text. Our starting point is the master equation governing the evolution of the state $\rho(t)$ of the DOMPO, Eq. (1), which, for convenience, we rewrite here as

$$\partial_t \rho = \mathcal{L}_{\rm opt}[\rho] + \mathcal{L}_m[\rho] + \mathcal{L}_{\rm OM}[\rho], \qquad (A1)$$

with

$$\mathcal{L}_{\rm opt}[\rho] = \left[-i\Delta_s a_s^\dagger a_s + \left(\epsilon_p a_p^\dagger + \frac{\chi}{2} a_p a_s^{\dagger 2} - {\rm H.c.} \right), \rho \right]$$
$$+ \gamma_0 \mathcal{D}_{a_p}[\rho] + \gamma_0 \mathcal{D}_{a_s}[\rho], \qquad (A2a)$$

$$\mathcal{L}_m[\rho] = [-i\Omega_m a_m^\dagger a_m, \rho] + \gamma_m (\bar{n}_{\rm th} + 1) \mathcal{D}_{a_m}[\rho]$$
$$+ \gamma_m \bar{n}_{\rm th} \mathcal{D}_{a_m^\dagger}[\rho], \qquad (A2b)$$

$$\mathcal{L}_{\rm OM}[\rho] = [i\Omega_m \eta_{\rm OM} a_s^\dagger a_s (a_m + a_m^\dagger), \rho]. \qquad (A2c)$$

All the quantities are defined in the text, and we recall the notation $\mathcal{D}_J[\rho] = 2J\rho J^\dagger - J^\dagger J\rho - \rho J^\dagger J$ for superoperators in Lindblad form.

In order to eliminate the optical modes and find an effective master equation for the mechanical state $\rho_m(t)$, we proceed as follows. We first define the projector superoperator $\mathcal{P}[\cdot] = \bar{\rho}_{\rm opt} \otimes {\rm tr}_{\rm opt}\{\cdot\}$, whose action on the full state $\rho(t)$ of the DOMPO is $\mathcal{P}[\rho(t)] = \bar{\rho}_{\rm opt} \otimes \rho_m(t)$. Here, $\bar{\rho}_{\rm opt}$ is the steady state of the optical Liouvillian, that is, $\mathcal{L}_{\rm opt}[\bar{\rho}_{\rm opt}] = 0$. Applying this superoperator and its complement $1 - \mathcal{P}$ to the master equation, and formally integrating the latter, we obtain an exact equation of motion for $\rho_m(t)$, the so-called *Nakajima-Zwanzig* equation [37,38]. This equation is not solvable, and therefore we apply a Born approximation, which takes into account terms up to second order in the optomechanical interaction. The resulting equation reads

$$\dot{\rho}_m(t) = \mathcal{L}_m \rho_m(t) + i\Omega_m \eta_{\rm OM} \bar{N}_s [x_m, \rho_m(t)]$$
$$- \Omega_m^2 \eta_{\rm OM}^2 \left[x_m, \int_0^t d\tau\, e^{\mathcal{L}_m \tau} [x_m \rho_m(t-\tau)s(\tau) - {\rm H.c.}] \right],$$
$$(A3)$$

where we have defined the mechanical position quadrature, $x_m = a_m + a_m^\dagger$, the photon number in the signal mode $\bar{N}_s = {\rm tr}\{a_s^\dagger a_s \bar{\rho}_{\rm opt}\}$, and the optical correlation function,

$$s(\tau) = {\rm tr}\{a_s^\dagger a_s e^{\mathcal{L}_{\rm opt}\tau}[a_s^\dagger a_s \bar{\rho}_{\rm opt}]\} - \bar{N}_s^2. \qquad (A4)$$

It is well known that the steady-state $\bar{\rho}_{\rm opt}$ of the DOPO is unique (which intuitively stems from the fact that both the pump and the signal modes have local dynamics leading to unique steady states, and the parametric interaction preserves that uniqueness), and hence $e^{\mathcal{L}_{\rm opt}\tau}$ is a relaxing map [43,44], mapping all optical operators O into the steady state, that is, $\lim_{\tau\to\infty} e^{\mathcal{L}_{\rm opt}\tau}[O] = {\rm tr}_{\rm opt}\{O\}\bar{\rho}_{\rm opt}$. Thus, the optical correlation function $s(\tau)$ will always decay to 0 within some finite memory time, which we denote $\tau_{\rm opt}$. Hence, in the asymptotic limit we can write $\lim_{t\to\infty} \rho_m(t-\tau) = \lim_{t\to\infty} \rho_m(t) \equiv \bar{\rho}_m$ in the integral kernel of Eq. (A3), obtaining an equation for $\bar{\rho}_m$ which is quadratic in the operators a_m and therefore allows for a Gaussian-state solution [45,46]. In other words, the equations for the first and second steady-state mechanical moments form a closed linear algebraic set,

$$0 = (-i\Omega_m - \gamma_m)\langle a_m \rangle + i\Omega_m \eta_{\rm OM} \bar{N}_s - \Omega_m^2 \eta_{\rm OM}^2 {\rm Re}\{d_0\}\langle x_m \rangle,$$
$$(A5a)$$

$$0 = \gamma_m(\bar{n}_{\rm th} - \langle \delta a_m^\dagger \delta a_m \rangle) - \Omega_m^2 \eta_{\rm OM} {\rm Re}\{(d_+ - d_-)\langle \delta a_m^\dagger \delta a_m \rangle$$
$$+ (d_-^* - d_+)\langle \delta a_m^2 \rangle - d_-\}, \qquad (A5b)$$

$$0 = (-i\Omega_m - \gamma_m)\langle \delta a_m^2 \rangle - \Omega_m^2 \eta_{\rm OM}^2 [(d_- - d_+^*)\langle \delta a_m^\dagger \delta a_m \rangle$$
$$+ (d_+ - d_-^*)\langle \delta a_m^2 \rangle + d_-], \qquad (A5c)$$

where we have used the abbreviations $\langle A \rangle = {\rm tr}\{A\bar{\rho}_m\}$, $\delta A = A - \langle A \rangle$, and

$$d_0 = \int_0^\infty d\tau\, s(\tau), \qquad (A6a)$$

$$d_\pm = \int_0^\infty d\tau\, e^{(\pm i\Omega_m - \gamma_m)\tau} s(\tau). \qquad (A6b)$$

These equations can be solved for the steady-state moments as functions of the optical photon number $\bar{N}_s$ and correlation function $s(\tau)$ without the need for further approximations. However, in order to obtain more physical insight into the mechanical steady-state $\bar{\rho}_m$ we apply both the Markov approximation and a rotating-wave approximation to Eq. (A3). The Markov approximation is based on the assumption that within the optical memory time $\tau_{\rm opt}$ all the mechanical dynamics can be neglected except for the evolution provided by the free Hamiltonian $\Omega_m a_m^{\dagger} a_m$. As a result, we can write $e^{\mathcal{L}_m \tau}[x_m \rho_m(t-\tau)] \approx x_m(\tau)\rho_m(t)$ with $x_m(\tau) = e^{i\Omega_m \tau} a_m + e^{-i\Omega_m \tau} a_m^{\dagger}$. On the other hand, the rotating-wave approximation consists of neglecting all the terms proportional to a_m^2 and $a_m^{\dagger 2}$ in the effective mechanical master equation, under the assumption that their rotation at frequency $2\Omega_m$ is much larger than the rates they are weighted by. After applying these approximations in Eq. (A3) we are left with an effective mechanical master equation given by

$$\dot{\rho}_m = \mathcal{L}_m \rho_m + i\Omega_m \eta_{\rm OM} \bar{N}_s [x_m, \rho_m]$$
$$+ \gamma_m \Gamma_- \mathcal{D}_{a_m}[\rho_m] + \gamma_m \Gamma_+ \mathcal{D}_{a_m^{\dagger}}[\rho_m], \quad (A7)$$

where the heating and cooling rates $\gamma_m \Gamma_{\pm} = \Omega_m^2 \eta_{\rm OM}^2$ $\mathrm{Re}\{d_{\mp}|_{\gamma_m=0}\}$ coincide precisely with those defined in the text. This master equation has a very simple Gaussian steady-state $\bar{\rho}_m$ corresponding to a displaced thermal state with mean

$$\langle a_m \rangle = \frac{i\Omega_m \eta_{\rm OM} \bar{N}_s}{i\Omega_m + \gamma_m} \approx \eta_{\rm OM} \bar{N}_s, \quad (A8)$$

phonon number $\langle \delta a_m^{\dagger} \delta a_m \rangle = \bar{n}_m$, where $\bar{n}_m$ is given by Eq. (4) in the text, and $\langle \delta a_m^2 \rangle = 0$. We note that, starting from a thermal state, the mechanical mode relaxes to this steady state at a rate $\gamma_{\rm eff} = \gamma_m (1 + \Gamma)$, where $\Gamma = \Gamma_- - \Gamma_+$ is what we call the cooling efficiency in the text, since the equations of motion for the phonon number fluctuations and the mechanical-field amplitude are given by

$$\partial_t \langle \delta a_m^{\dagger} \delta a_m \rangle = -2\gamma_m(1+\Gamma)\langle \delta a_m^{\dagger} \delta a_m \rangle + 2\gamma_m(\bar{n}_{\rm th} + \Gamma_-),$$
$$\partial_t \langle a_m \rangle = [-i\Omega_m - \gamma_m(1+\Gamma)]\langle a_m \rangle + i\eta_{\rm OM}\Omega_m \langle a_s^{\dagger} a_s \rangle. \quad (A9)$$

We have checked that this rate $\gamma_{\rm eff}$ is lower than the decay rate of the optical correlator $s(\tau)$ for the parameters of interest, hence making the Markov a valid approximation.

Let us remark that throughout the work we have been using both Eqs. (A3) and (A7) to obtain the steady-state moments of the mechanical oscillator. We have never observed any notable differences between them, except when working extremely close to threshold within the semiclassical approach (see the inset in Fig. 3). In these cases, however, the failure of Eq. (A7) can be directly attributed to the failure of the semiclassical approach, and not to the failure of the rotating-wave approximation itself, which indeed is very well satisfied as shown by the c-MoP approach. Thus, we conclude that for the parameter regime studied in this work (compatible with current cWGM resonators) the state of the mechanical oscillator is indeed a displaced thermal state, with a phonon number that can only be evaluated once the optical photon number $\bar{N}_s$ and correlation function $s(\tau)$ are known.

APPENDIX B: SEMICLASSICAL APPROACH

The simplest way to obtain the optical correlator $s(\tau)$ is by using standard linearization on $\mathcal{L}_{\rm opt}$. In this approach, we move to a displaced picture in which the large coherent background of the pump mode is removed and then keep the terms of the transformed optical Liouvillian only up to second order in the bosonic operators. The displacement operator $D = \exp[\epsilon_p(a_p - a_p^{\dagger})/\gamma_0]$ allows us to move to the new picture, in which the transformed optical state $\tilde{\rho}_{\rm opt} = D^{\dagger}\rho_{\rm opt}D$ evolves according to a transformed Liouvillian, $\tilde{\mathcal{L}}_{\rm opt} = D^{\dagger}\mathcal{L}_{\rm opt}D$. Removing terms beyond quadratic order, this transformed Liouvillian can be written as the sum of independent Liouvillians for the pump and signal modes, $\tilde{\mathcal{L}}_{\rm opt} = \mathcal{L}_p + \mathcal{L}_s$, with $\mathcal{L}_p = \gamma_0 \mathcal{D}_{a_p}$ and

$$\gamma_0^{-1}\mathcal{L}_s(\cdot) = \left[-i\Delta a_s^{\dagger} a_s + \frac{\sigma}{2}(a_s^{\dagger 2} - a_s^2), \cdot\right] + \mathcal{D}_{a_s}[\cdot], \quad (B1)$$

with the injection parameter $\sigma = \epsilon_p \chi/\gamma_0^2$ and normalized detuning $\Delta = \Delta_s/\gamma_0$. Consequently, the optical steady state in the original picture becomes the separable state $\bar{\rho}_{\rm opt} = |\epsilon_p/\gamma_0\rangle\langle\epsilon_p/\gamma_0| \otimes \bar{\rho}_s$, where $|\epsilon_p/\gamma_0\rangle$ is a coherent state of amplitude ϵ_p/γ_0 and $\bar{\rho}_s$ is the Gaussian state satisfying $\mathcal{L}_s[\bar{\rho}_s] = 0$. The latter is completely characterized by its first and second moments, which are trivially found to be $\langle a_s \rangle = 0$, $\langle a_s^{\dagger} a_s \rangle = \sigma^2/2(1 + \Delta^2 - \sigma^2) \equiv \bar{N}_s$ and $\langle a_s^2 \rangle = \sigma(1 - i\Delta)/2(1 + \Delta^2 - \sigma^2)$, where we use the usual notation $\langle A \rangle = \mathrm{tr}_s\{A\rho_s\}$ for any operator A acting on the signal subspace.

The optical correlation function simplifies to $s(\tau) = \mathrm{tr}\{a_s^{\dagger} a_s e^{\mathcal{L}_s \tau} \mu_s\}$, where we have defined a traceless operator, $\mu_s = (a_s^{\dagger} a_s - \bar{N}_s)\bar{\rho}_s$. Using again the fact that the Liouvillian $\mathcal{L}_s$ is Gaussian, it is simple to evaluate the correlation function $s(\tau)$. Toward this aim, let us define the column vector

$$\vec{v}(\tau) = \mathrm{col}\left(\widetilde{\langle a_s^{\dagger} a_s \rangle}, \widetilde{\langle a_s^2 \rangle}, \widetilde{\langle a_s^{\dagger 2} \rangle}\right), \quad (B2)$$

where the expectation value of an operator A with a tilde is defined as $\widetilde{\langle A \rangle} = \mathrm{tr}\{Ae^{\mathcal{L}_s \tau}\mu_s\}$. Taking the derivative of this vector with respect to τ, we find the linear system $\partial_\tau \vec{v}(\tau) = L\vec{v}(\tau)$, where the matrix M reads

$$L = \gamma_0 \begin{pmatrix} -2 & \sigma & \sigma \\ 2\sigma & -2(1 + i\Delta) & 0 \\ 2\sigma & 0 & -2(1 - i\Delta) \end{pmatrix}. \quad (B3)$$

It is straightforward to solve this linear system, for example, by diagonalizing L. We write $L = U\Lambda U^{-1}$, with a similarity matrix U that can be found analytically (but its expression is too lengthy to be reported here) and a diagonal matrix Λ containing the eigenvalues of L, $\lambda_1 = -2\gamma_0$, and $\lambda_{2,3} = -2\gamma_0(1 \pm i\sqrt{\Delta^2 - \sigma^2})$. Note that for $\sigma > \Delta$ the square root becomes imaginary, making $\lambda_2 < \gamma_0$, and in fact $\lambda_2 = 0$ at threshold, $\sigma = \sqrt{1 + \Delta^2}$. Consequently, we call the region with $\sigma > \Delta$ the *critical slowing-down regime*. The solution of the linear system is then found as

$$\vec{v}(\tau) = Ue^{\Lambda\tau}U^{-1}\vec{v}(0) \equiv \sum_{n=1}^{3} L_n e^{\lambda_n \tau}\vec{u}, \quad (B4)$$

where we have defined the initial condition vector

$$\vec{u} = \vec{v}(0) = \text{col}\big(\langle a_s^\dagger a_s a_s^\dagger a_s\rangle - \bar{N}_s^2, \langle a_s^2 a_s^\dagger a_s\rangle$$
$$- \langle a_s^2\rangle \bar{N}_s, \langle a_s^{\dagger 3} a_s\rangle - \langle a_s^2\rangle^* \bar{N}_s\big), \tag{B5}$$

and the matrices $L_n = U\Pi_n U^{-1}$, where $(\Pi_n)_{jl} = \delta_{jn}\delta_{ln}$. Note that the vector $\vec{u}$ is formed by fourth-order moments. In order to find them, we simply exploit the Gaussian structure of $\mathcal{L}_s$, which allows us to express moments of any order as products of moments of first and second order. Specifically, concerning third- and fourth-order moments we simply use

$$\langle \delta a_s^\dagger \delta a_s^2\rangle = \langle \delta a_s^3\rangle = 0, \tag{B6a}$$

$$\langle \delta a_s^{\dagger 2} \delta a_s^2\rangle = \langle \delta a_s^{\dagger 2}\rangle\langle \delta a_s^2\rangle + 2\langle \delta a_s^\dagger \delta a_s\rangle^2, \tag{B6b}$$

$$\langle \delta a_s^\dagger \delta a_s^3\rangle = 3\langle \delta a_s^\dagger \delta a_s\rangle\langle \delta a_s^2\rangle, \tag{B6c}$$

where $\delta a_s = a_s - \langle a_s\rangle$. Note that the optical correlation function we are looking for is given by the first component of the vector, $s(\tau) = [\vec{v}(\tau)]_1$, and the integrals appearing in d_0 and $d_\pm$ in Eq. (A6) can be easily evaluated due to the exponential τ dependence of $\vec{v}(\tau)$ in Eq. (B4).

APPENDIX C: COOLING BY HEATING VIA A TWO-PHOTON PROCESS

In the text we have shown that significant cooling can be obtained when working in the resolved sideband regime $\Omega = \Omega_m/\gamma_0 \gg 1$ and close to the resonance condition $\delta_{\text{eff}} = \sqrt{\Delta^2 - \sigma^2} = \Omega/2$. We have interpreted this phenomenon as a cooling-by heating effect, for which we have moved to a new picture defined by the squeezing operator $S(r) = \exp[-ir(a_s^{\dagger 2} + a_s^2)/2]$ within the semiclassical approach explained above. Here we want to explicitly perform an adiabatic elimination of the optical mode in this picture, which will allow us to gain more insight into the cooling mechanism.

Let us first write the master equation in this "squeezed picture." The transformation diagonalizes the Hamiltonian in the optical Liouvillian, (B1), turning it into

$$\gamma_0^{-1} S^\dagger(r)\mathcal{L}_{\text{opt}}S(r)[\cdot] = i[\delta_{\text{eff}}a_s^\dagger a_s, \cdot] + (1 + \bar{N}_{\text{eff}})\mathcal{D}_{a_s}[\cdot]$$
$$+ \bar{N}_{\text{eff}}\mathcal{D}_{a_s^\dagger}[\cdot] + iM\mathcal{K}_{a_s}[\cdot] - iM\mathcal{K}_{a_s^\dagger}[\cdot], \tag{C1}$$

where we have defined the superoperator $\mathcal{K}_J[\cdot] = 2J(\cdot)J - J^2(\cdot) - (\cdot)J^2$, as well as the parameters $\bar{N}_{\text{eff}} = (\Delta/\delta_{\text{eff}} - 1)/2$ and $M = \sigma/2\delta_{\text{eff}}$. Note that the $\mathcal{K}$ terms rotate at frequency $2\delta_{\text{eff}}$ and, hence, are highly suppressed when we work within the cooling condition $\delta_{\text{eff}} \approx \Omega/2$ and with $M/2\delta_{\text{eff}} = \sigma/4\delta_{\text{eff}}^2 \ll 1$ (rotating-wave approximation). Therefore, as mentioned in the text, in this picture the signal field is turned into a bosonic mode with oscillation frequency δ_{eff} and at thermal equilibrium with occupation $\bar{N}_{\text{eff}}$. On the other hand, the photon-number operator is transformed into $S^\dagger(r)a_s^\dagger a_s S(r) = \bar{N}_{\text{eff}} + (2\bar{N}_{\text{eff}} + 1)a_s^\dagger a_s + iM(a_s^2 - a_s^{\dagger 2})$, and hence the optomechanical interaction can be approximated by

$$S^\dagger(r)a_s^\dagger a_s S(r)(a_m + a_m^\dagger) \approx iM\big(a_s^2 a_m^\dagger - a_s^{\dagger 2} a_m\big) \tag{C2}$$

within the rotating-wave approximation as long as $\eta_{\text{OM}}(2\bar{N}_{\text{eff}} + 1) = \eta_{\text{OM}}\Delta/\delta_{\text{eff}} \ll 1$.

Hence, within these conditions, the transformed state $\tilde{\rho} = S^\dagger(r)\rho S(r)$ evolves according to a master equation that we write as

$$\partial_t\tilde{\rho} = \tilde{\mathcal{L}}_s[\tilde{\rho}] + \mathcal{L}_m[\tilde{\rho}] + \tilde{\mathcal{L}}_{\text{OM}}[\tilde{\rho}], \tag{C3}$$

with

$$\tilde{\mathcal{L}}_s[\tilde{\rho}] = [-i\gamma_0\delta_{\text{eff}}a_s^\dagger a_s, \tilde{\rho}] + \gamma_0(\bar{N}_{\text{eff}} + 1)\mathcal{D}_{a_s}[\tilde{\rho}]$$
$$+ \gamma_0\bar{N}_{\text{eff}}\mathcal{D}_{a_s^\dagger}[\tilde{\rho}], \tag{C4a}$$

$$\mathcal{L}_m[\tilde{\rho}] = [-i\Omega_m a_m^\dagger a_m, \tilde{\rho}] + \gamma_m(\bar{n}_{\text{th}} + 1)\mathcal{D}_{a_m}[\tilde{\rho}]$$
$$+ \gamma_m\bar{n}_{\text{th}}\mathcal{D}_{a_m^\dagger}[\tilde{\rho}], \tag{C4b}$$

$$\tilde{\mathcal{L}}_{\text{OM}}[\tilde{\rho}] = \big[\Omega_m\eta_{\text{OM}}M\big(a_s^2 a_m^\dagger - a_s^{\dagger 2} a_m\big), \tilde{\rho}\big], \tag{C4c}$$

where $\bar{N}_{\text{eff}}$ and M are defined in the text. The structure of this master equation is similar to the original one, Eq. (A1), with the only difference that the optical Liouvillian is replaced by $\tilde{\mathcal{L}}_s$, corresponding to a single mode at finite temperature, and the optomechanical interaction $a_s^\dagger a_s(a_m + a_m^\dagger)$, by $iM(a_s^2 a_m^\dagger - a_s^{\dagger 2} a_m)$. The adiabatic elimination of the optical mode can be carried out in exactly the same way as in Appendix A, and under the cooling condition $\delta_{\text{eff}} = \Omega/2$ it would lead to the heating and cooling rates

$$\Gamma_- \approx \tfrac{1}{2}CM^2\text{tr}\big\{a_s^2 a_s^{\dagger 2}\tilde{\rho}_s\big\}, \tag{C5a}$$

$$\Gamma_+ \approx \tfrac{1}{2}CM^2\text{tr}\big\{a_s^{\dagger 2} a_s^2\tilde{\rho}_s\big\}, \tag{C5b}$$

where $C = \Omega_m^2\eta_{\text{OM}}^2/\gamma_m\gamma_0$ is the bare optomechanical cooperativity, and $\tilde{\rho}_s$ is a thermal state with mean photon number $\bar{N}_{\text{eff}}$. The cooling efficiency is then given by

$$\Gamma = \Gamma_- - \Gamma_+ = \tfrac{1}{2}CM^2\text{tr}\big\{[a_s^2, a_s^{\dagger 2}]\tilde{\rho}_s\big\} = 2CM^2\big(\bar{N}_{\text{eff}} + \tfrac{1}{2}\big). \tag{C6}$$

The cooling-by-heating effect is clearly seen because the cooling efficiency increases with the effective thermal photon number $\bar{N}_{\text{eff}}$. But it is important to note that this enhancement of the cooling efficiency is a direct consequence of the commutator appearing in the trace, contributing as $[a_s^2, a_s^{\dagger 2}] = 4a_s^\dagger a_s + 2$, which in turn comes from the fact that the effective optomechanical interaction $i(a_s^2 a_m^\dagger - a_s^{\dagger 2} a_m)$ corresponds to the exchange of phonons with pairs of photons. In the usual sideband laser-cooling scenario, the effective optomechanical interaction is bilinear, e.g., $i(a_s a_m^\dagger - a_s^\dagger a_m)$, meaning that the commutator in the expression above is replaced by $[a_s, a_s^\dagger] = 1$, and hence the thermal photonic background does not enter the cooling efficiency.

Let us, finally, note that the fundamental limit can be written as

$$\bar{n}_{\text{FL}} = \frac{\Gamma_+}{\Gamma} = \frac{\text{tr}\big\{a_s^{\dagger 2} a_s^2\tilde{\rho}_s\big\}}{\text{tr}\big\{[a_s^2, a_s^{\dagger 2}]\tilde{\rho}_s\big\}} = \frac{\bar{N}_{\text{eff}}^2}{2\bar{N}_{\text{eff}} + 1} \xrightarrow{\bar{N}_{\text{eff}} \gg 1} \frac{\bar{N}_{\text{eff}}}{2}, \tag{C7}$$

which increases linearly with the effective thermal photon number. Hence, as explained in the text, the cooling-by-heating mechanism is optimized by finding a proper trade-off between the increase in the cooling efficiency (good noise) and the

increase in the fundamental limit (bad noise). It is to be noted that within the usual sideband laser cooling, any thermal background will still contribute to this fundamental limit, $\bar{n}_{\rm FL} = {\rm tr}\{a_s^\dagger a_s \rho_s\}/{\rm tr}\{[a_s, a_s^\dagger]\rho_s\} \xrightarrow{\bar{N}_{\rm eff} \gg 1} \bar{N}_{\rm eff}$, but as explained above, it provides no enhancement of the cooling efficiency Γ. In other words, in standard sideband cooling the thermal background acts only as "bad" noise.

We emphasize that these expressions for Γ and $\bar{n}_{\rm FL}$ agree with the ones provided in the text, which were first calculated exactly within the semiclassical approach and then approximated to leading order in $1/\Omega^2$ for $\delta_{\rm eff} = \Omega/2$.

APPENDIX D: c-MoP APPROACH TO THE OPTICAL PROBLEM AND SCALINGS AT THE CRITICAL POINT

Despite the analytical and physical insight that it provides, the semiclassical approach suffers from several issues, most importantly its divergent character at threshold, which shows that it cannot be trusted when working close to this point. Unfortunately, there is no systematic way of checking within the formalism itself where exactly it fails. This question can only be answered by comparing it to a more accurate approach. Toward this aim, we have applied our recently developed c-MoP theory [24,25]. In particular, this approach has allowed us to characterize optical steady-state observables such as the photon number $\bar{N}_s$ and the correlation function $s(\tau)$ in all relevant parameter space, including the threshold.

We have detailed the application of c-MoP theory [24] to the DOPO problem in [25], including its combination with the Gaussian-state approximation that we use in this work, which was shown to be quite accurate for both steady-state quantities and dynamics. Let us now briefly introduce this approach here for completeness, keeping in mind that details can be looked up in [25]. Our main goal consists of finding the optical correlation function, (5), for which we need to solve the dynamics generated by the optical Liouvillian $\mathcal{L}_{\rm opt}$, that is, the DOPO dynamics,

$$\dot{\rho}_{\rm opt} = \mathcal{L}_{\rm opt}\rho_{\rm opt}. \tag{D1}$$

This is a nonlinear two-mode problem with no analytic solution and whose direct numerical simulation becomes unfeasible already for moderate photon numbers. In contrast, c-MoP theory works with the reduced states of the pump and signal modes, $\rho_p(t) = {\rm tr}_s\{\rho_{\rm opt}\}$ and $\rho_s(t) = {\rm tr}_p\{\rho_{\rm opt}\}$, respectively, whose coupled dynamics are approximated by the set of nonlinearly coupled equations,

$$\dot{\rho}_s(t) = \mathcal{L}_s\rho_s(t) + \frac{\chi}{2}\left[a_s^{\dagger 2}\langle a_p\rangle(t) - a_s^2\langle a_p^\dagger\rangle(t), \rho_s(t)\right]$$

$$+ \left(\frac{\chi}{2}\right)^2\{[a_s^2, h_s(t)] + {\rm H.c.}\}, \tag{D2a}$$

$$\dot{\rho}_p(t) = \mathcal{L}_p\rho_p(t) + \frac{\chi}{2}\left[a_p\langle a_s^{\dagger 2}\rangle(t) - a_p^\dagger\langle a_s^2\rangle(t), \rho_p(t)\right]$$

$$+ \left(\frac{\chi}{2}\right)^2\left\{\left[a_p, \sum_{n=1}^3 h_{p,n}(t)\right] + {\rm H.c.}\right\}, \tag{D2b}$$

$$\dot{h}_s(t) = (-\gamma_p + \mathcal{L}_s)h_s(t) + \mathcal{K}_s(t,t)\rho_s(t), \tag{D2c}$$

$$\dot{h}_{p,n}(t) = (\lambda_n + \mathcal{L}_p)h_{p,n}(t) + \mathcal{K}_{p,n}(t,t)\rho_p(t), \tag{D2d}$$

where $h_s(t)$ and $\{h_{p,n}(t)\}_{n=1,2,3}$ are auxiliary operators acting on the signal and pump subspaces (introduced to turn the c-MoP equations into ordinary differential equations, since originally they have an integrodifferential structure [25]), and we refer to [25] for the definitions of the superoperators $\mathcal{L}_s$, $\mathcal{L}_p$, $\mathcal{K}_s(t,t')$, and $\mathcal{K}_{p,n}(t,t')$. Denoting by D_s and D_p the dimensions of the truncated Hilbert spaces of the signal and pump in a numerical simulation, we see that the original problem, (D1), requires solving a $D_s \times D_p$ system, whereas there are only $2D_s + 4D_p$ c-MoP equations (or even fewer if the quadratic or Gaussian structure of the pump equations is exploited).

Nevertheless, even though c-MoP allows us to gain numerical insight into a larger region of parameter space, the simulation of problems with very large photon numbers (such as the ones we work with close to threshold) is still challenging. It is in these regions where a Gaussian-state approximation becomes extremely useful. As the name suggests, this approximation consists of assuming that the reduced signal and pump states are Gaussian, meaning that they are completely characterized by first- and second-order moments. Under such circumstances, we can approximate third- and fourth-order moments of $\bar{\rho}_s$ as in (B6), and the c-MoP equations provide a closed set of nonlinear equations for the first- and second-order moments of the operators $\rho_s(t)$, $\rho_p(t)$, $h_s(t)$, and $\{h_{p,n}(t)\}_{n=1,2,3}$. The steady-state moments can then be efficiently found simply by finding the stationary solutions of these equations.

As an example, in Fig. 4(a) we show the steady-state photon number $\bar{N}_s$ at the critical point ($x = 1$) as a function of the normalized detuning Δ. It shows a clear linear dependence on $\Delta > 1$, which, together with the well-known $\eta_{\rm DC}^{-1}$ scaling with the down-conversion coupling [25,47–49], provides an overall $\bar{N}_s \propto (1 + \Delta)/\eta_{\rm DC}$ scaling of the signal photon number at threshold. The knowledge of this scaling plays an important role in the determination of the conditions under which mechanical backaction on the optics can be neglected, as explained in Sec. VI.

Let us now explain how the optical correlation function can be evaluated within this framework. First, note that we can rewrite it as

$$s(\tau) = \bar{N}_s \left({\rm tr}\{a_s^\dagger a_s \nu(\tau)\} - \bar{N}_s\right), \tag{D3}$$

where $\nu(t) = e^{\mathcal{L}_{\rm opt}t}\nu(0)$ can be interpreted as an operator with evolution equation $\dot{\nu} = \mathcal{L}_{\rm opt}\nu$ and initial condition $\nu(0) = a_s^\dagger a_s \bar{\rho}_{\rm opt}/\bar{N}_s$. Since this evolution equation is formally equivalent to the optical master equation, (D1), we can apply c-MoP theory directly to $\nu(t)$, approximating it by a separable operator $\nu_s(t) \otimes \nu_p(t)$, with $\nu_p(t) = {\rm tr}_s\{\nu(t)\}$ and $\nu_s(t) = {\rm tr}_p\{\nu(t)\}$ evolving according to Eqs. (D2) with ρ_j replaced by ν_j. Under a Gaussian approximation for $\nu_s(t)$ similar to (B6) but with expectation values defined with respect to $\nu_s(t)$, the evolution equations for the first and second moments of $\nu_s(t)$, $\nu_p(t)$, $h_s(t)$, $\{h_{p,n}(t)\}_{n=1,2,3}$, and their Hermitian conjugates (note that ν is not Hermitian) form a closed nonlinear system which we can solve again efficiently. Note that the initial conditions for these moments, e.g., ${\rm tr}\{a_s^\dagger a_s \nu(0)\} = {\rm tr}\{a_s^\dagger a_s a_s^\dagger a_s \bar{\rho}_{\rm opt}\}/\bar{N}_s$, are found from the Gaussian c-MoP steady-state solutions as explained above.
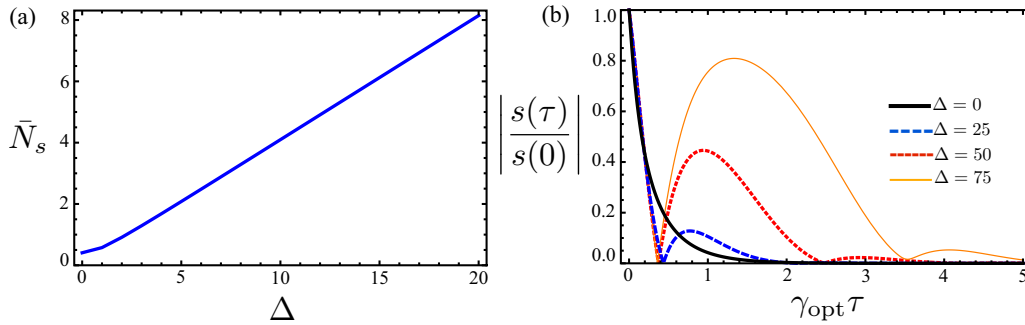
FIG. 4. (a) Steady-state photon number $\bar{N}_s = \mathrm{tr}\{a_s^\dagger a_s \bar{\rho}_{\mathrm{opt}}\}$ at the critical point as a function of the normalized detuning Δ. (b) Absolute value of the normalized optical correlation function $s(\tau)$ at the critical point for different values of detuning as of function of the normalized time $\gamma_{\mathrm{opt}}\tau$, with $\gamma_{\mathrm{opt}} = \gamma_0 \eta_{\mathrm{DC}}(1 + \Delta)$. In both panels the parameters are $\gamma_0 = 1$ and $\eta_{\mathrm{DC}} = 0.01$, and we obtained them by applying Gaussian c-MoP theory to the optical problem.

In Fig. 4(b) we show the evolution of the absolute value of the correlation function $s(\tau)$ at the critical point and for different values of the normalized detuning Δ. Time is normalized to $[\gamma_0 \eta_{\mathrm{DC}}(1 + \Delta)]^{-1}$, and hence the fact that all

the curves decay on the same time scale proves that the optical relaxation time scales as $\gamma_{\mathrm{opt}} = \gamma_0 \eta_{\mathrm{DC}}(1 + \Delta)$ at threshold. This again plays a fundamental role when proving that mechanical backaction is negligible, as discussed in Sec. VI.

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