Inducing Nonclassical Lasing via Periodic Drivings in Circuit Quantum Electrodynamics

Carlos Navarrete-Benlloch,[1] Juan José García-Ripoll,[2] and Diego Porras[3,4]
[1]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-strasse 1, 85748 Garching, Germany*
[2]*Instituto de Física Fundamental, IFF-CSIC, Serrano 113-bis, Madrid E-28006, Spain*
[3]*Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN19QH, United Kingdom*
[4]*Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain*
(Received 7 February 2014; published 7 November 2014)

We show how a pair of superconducting qubits coupled to a microwave cavity mode can be used to engineer a single-atom laser that emits light into a nonclassical state. Our scheme relies on the dressing of the qubit-field coupling by periodic modulations of the qubit energy. In the dressed basis, the radiative decay of the first qubit becomes an effective incoherent pumping mechanism that injects energy into the system, hence turning dissipation to our advantage. A second, auxiliary qubit is used to shape the decay within the cavity, in such a way that lasing occurs in a squeezed basis of the cavity mode. We characterize the system both by mean-field theory and exact calculations. Our work may find applications in the generation of squeezing and entanglement in circuit QED, as well as in the study of dissipative few- and many-body phase transitions.

DOI: 10.1103/PhysRevLett.113.193601 PACS numbers: 42.50.Pq, 42.50.Dv, 42.55.-f, 85.25.-j

Introduction.—Recent progress in experimental solid-state quantum optics has led to exciting possibilities for the control of quantum states of the electromagnetic field. Circuit quantum electrodynamics (QED) [1,2] is one of such new platforms and can be seen as the microwave counterpart of cavity QED, with optical cavities and atoms replaced, respectively, by linear and nonlinear superconducting circuits. The latter are usually referred to as "artificial atoms" or "superconducting qubits." In circuit QED single emitters are placed permanently, and different quantum-optical elements can be combined by fabrication. The field emitted by those devices can be integrated into circuits in the form of itinerant fields and, hence, new ideas for generating quantum photonic states are of major importance for applications of this emerging field.

In recent years various experiments have shown that single-qubit lasing is possible in this scenario [3,4], while at the same time the generation of squeezed fields via Josephson parametric amplifiers has taken a lot of attention [5–7]. Here we propose a scheme that combines these two physical scenarios, motivated by two main advantages that circuit QED offers with respect to their optical counterparts. (a) Superconducting qubits having frequencies within the microwave range, the system parameters can be modulated with rates and amplitudes comparable to their characteristic energy, which opens the way to a versatile control of the qubit-field couplings via periodic drivings [8], something that in optical implementations typically involves Raman transitions which rely on the atomic internal structure [9]. So far this has allowed for the observation of the dynamical Casimir effect [10,11], as well as motivated proposals for the simulation of the ultrastrong coupling regime of quantum optics [12]. (b) Several cavities and dissipative elements can be permanently coupled to single qubits. Thus, they provide us with an ideal toolbox for engineering dissipative processes [13] that are very challenging to implement in atomic QED.

We exploit these advantages for the design of a *lasing dissipative phase transition* in which light is emitted into a *squeezed state*, that is, a *nonclassical state* in the sense of Glauber [14,15]. In particular, we show the following. (i) A periodic driving of the qubit energy is able to induce an effective counterrotating-type interaction with the field, which turns the qubit relaxation into an effective population inversion mechanism, hence turning dissipation into something useful. This leads to single-atom lasing into a classical, coherent state. (ii) A biperiodic driving allows us to shape the qubit-field interaction such that photons are emitted into a squeezed photonic mode. A mean-field description of this problem allows us to predict a lasing transition. Surprisingly, if decay occurs by normal cavity leakage, dissipation still drives the system into a classical lasing phase. (iii) An additional qubit can be used to induce a cavity decay mechanism that cools it into a squeezed vacuum [8,16]. The joint action of this cooling process and the emission of light into a squeezed mode yields *lasing into a squeezed state*. (iv) Our ideas can be implemented in circuit QED setups with state-of-the-art experimental parameters, thus leading to a scheme that goes beyond single-atom lasing into coherent states in atomic [17] or solid-state [18–22] systems.

In addition to applications related to bright sources of squeezed or entangled light, the scalability of our scheme paves the way to the study of dissipative phase transitions in mesoscopic lattice QED systems [23] since many-qubit extensions of our work [24] pose an intriguing many-body problem where strongly correlated phenomena could be analyzed.

Single artificial atom and cavity system.—As shown in Fig. 1, we consider one mode of a cavity coupled to a qubit

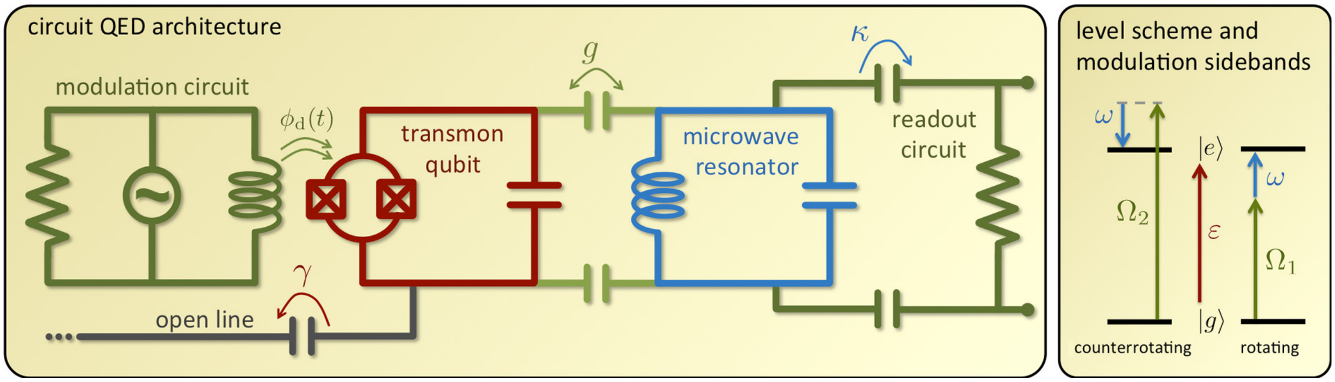

FIG. 1 (color online). (Left panel) Circuit QED architecture of the system: A superconducting qubit (transmon [25,26]) is capacitively coupled to an LC resonator of frequency ω, while its transition frequency between the ground $|g\rangle$ and excited $|e\rangle$ states is modulated via the flux generated by an external circuit; the qubit is coupled to an open transmission line to which it can radiate excitations, while the resonator experiences leakage to a readout circuit. (Right panel) A biperiodic modulation with frequencies matching the lower and upper sidebands of the qubit-resonator system allows us to independently tune the relative amplitudes of the rotating and counterrotating terms of the qubit-field interaction.

whose transition frequency is modulated in time. Such a system is described by a time-dependent Hamiltonian $H(t) = H_0 + H_{\rm int} + H_{\rm d}(t)$, with $H_0 = \omega a^\dagger a + \epsilon\sigma_z/2$, $H_{\rm int} = g(a + a^\dagger)\sigma_x$, and $H_{\rm d}(t) = \sum_{j=1}^{n_{\rm d}} \Omega_j \eta_j \cos(\Omega_j t)\sigma_z$, where we have set $\hbar = 1$ and have assumed that the modulation is multiperiodic. ω is the cavity frequency and a the corresponding annihilation operator, $\sigma_{z,x}$ are the Pauli operators associated to the qubit with bare frequency ε, and $H_{\rm d}(t)$ describes $n_{\rm d}$ periodic drivings with frequencies Ω_j and normalized amplitudes η_j.

Additionally, we consider two dissipative channels, one describing the radiative decay of the qubit to an open transmission line at rate γ and another for the cavity losses at rate κ. We will employ the notation $\mathcal{L}_{\{O,\Gamma\}}[\rho] = \Gamma(2O\rho O^\dagger - O^\dagger O\rho - \rho O^\dagger O)$, such that the master equation governing the evolution of the system's state ρ reads

$$\dot{\rho} = -i[H(t), \rho] + \mathcal{L}_{\{\sigma,\gamma\}}[\rho] + \mathcal{L}_{\{a,\kappa\}}[\rho]. \qquad (1)$$

We will be considering a far-off resonant and weak coupling regime ($g \ll \varepsilon, \omega, |\varepsilon - \omega|$), such that in the absence of driving the steady state corresponds to the trivial photon vacuum. However, we show below that by switching on an appropriate modulation $H_{\rm d}(t)$, energy can be injected into the system, driving it into a lasing regime.

Shaping the interaction.—Consider a bichromatic driving ($n_{\rm d} = 2$) modulating at the upper and lower sidebands, $\Omega_{1,2} = \varepsilon \mp \omega$; see Fig. 1. Moving to an interaction picture with respect to $H_0 + H_{\rm d}(t)$, we show in the Supplemental Material [27] that the system dynamics is well captured by the time-independent Hamiltonian $\tilde{H} = -\tilde{g}(ua^\dagger + va)\sigma^\dagger + \text{H.c}$; we have defined the parameters $u = J_0(2\eta_1)J_1(2\eta_2)/N$ and $v = J_0(2\eta_2)J_1(2\eta_1)/N$, which satisfy the Bogoliubov relation $|u^2 - v^2| = 1$ with the definition $N^2 = |J_0^2(2\eta_1)J_1^2(2\eta_2) - J_0^2(2\eta_2)J_1^2(2\eta_1)|$ being $J_m(z)$ the Bessel function of order m, as well as a renormalized coupling $\tilde{g} = gN$. Note that we are describing here the renormalization of the qubit-field interaction by photon-assisted tunneling in a nonperturbative regime with respect to the modulation amplitudes [28].

Hence, we see that a biperiodic modulation allows us to tune the relative weights of the rotating and counterrotating terms of the qubit-field interaction, which we exploit in the following to generate lasing to coherent or squeezed states.

Single-qubit lasing.—Consider first the simple case $\eta_1 = 0$, in which we drive the qubit with a single frequency ($u = 1, v = 0$). In this case the qubit is coupled to the cavity mode through a counterrotating-type interaction, so that the master equation of the system reads $\dot{\rho} = i\tilde{g}[a^\dagger\sigma^\dagger + a\sigma, \rho] + \mathcal{L}_{\{\sigma,\gamma\}}[\rho] + \mathcal{L}_{\{a,\kappa\}}[\rho]$. By using the transformation $\sigma \leftrightarrow \sigma^\dagger$ (a permutation of the qubit basis, irrelevant for the field dynamics), we see that the qubit relaxation is turned into an effective incoherent pump of its population together with a corotating coupling to the field. This leads to our first result: the periodic driving induces a lasing mechanism. This equation has been studied in the past [29–32], and mean-field theory predicts a lasing transition that depends on the cooperativity parameter $\tilde{C} = \tilde{g}^2/\gamma\kappa$. If $\tilde{C} \gg 1$ and the inversion rate is much faster than the cavity losses, $\gamma \gg \kappa$, the steady state of the cavity consists in a coherent state with a random phase.

Engineering nonclassical lasing.—Let us now consider the situation in which both driving amplitudes $\eta_{1,2}$ are nonzero, so that the qubit is coupled to a squeezed mode $A = ua + va^\dagger$ instead of the original cavity mode a. Choosing $|u| > |v|$, the interaction takes the form $\tilde{H} = -\tilde{g}(A^\dagger\sigma^\dagger + A\sigma)$. This seems to suggest lasing into the squeezed mode A, and thus emission of a bright squeezed state of light. However, we show below that a careful study of the master equation shows that this is not the case since losses still take place by photon decay in the original cavity mode basis, a, through the term $\mathcal{L}_{a,\kappa}$ in Eq. (1). We prove in the following that in order to achieve lasing in the squeezed mode, A, cavity decay has to occur on that basis. We thus introduce a second, auxiliary qubit that will be used to control the photon decay in the cavity, following the ideas introduced in [8]. We assume that the auxiliary qubit is controlled by the same driving parameters, except for an exchange of the amplitudes $\eta_1 \leftrightarrow \eta_2$ which makes $|u| < |v|$, such that one effectively generates the rotating-type interaction $\tilde{H}' = -\tilde{g}'(A^\dagger \sigma' + A\sigma'^\dagger)$, where σ' and $\tilde{g}'$ correspond to operators and couplings of the auxiliary qubit, respectively. The latter has a decay rate γ', such that in the limit $\gamma' \gg \tilde{g}'\sqrt{\langle a^\dagger a\rangle}$, it can be adiabatically eliminated. This way, we obtain the master equation,

$$\dot{\rho} = i\tilde{g}[A^\dagger\sigma^\dagger + A\sigma, \rho] + \mathcal{L}_{\{\sigma,\gamma\}}[\rho] + \mathcal{L}_{\{a,\kappa\}}[\rho] + \mathcal{L}_{\{A,\kappa\tilde{C}'\}}[\rho]. \quad (2)$$

If condition $\tilde{C}' = \tilde{g}'^2/\gamma'\kappa \gg v^2$ is met, the effective dissipator in the squeezed mode $\mathcal{L}_{\{A,\kappa\tilde{C}'\}}$ dominates the natural cavity dissipation $\mathcal{L}_{\{a,\kappa\}}$ and the system behaves as a laser for the squeezed mode A, and hence as a nonclassical laser for the original cavity mode a.

In order to get an approximate description of the steady state predicted by this master equation, we apply a mean-field approximation in which ρ is assumed to be separable in the qubit-field subspaces [30]. Defining the expectation values $F = \langle A\rangle$, $S = i\langle\sigma\rangle^*$, and $D = -\langle\sigma_z\rangle$, we get the nonlinear system of equations

$$\dot{F} = -\kappa_{\text{eff}}F + \tilde{g}S, \qquad \dot{S} = -\gamma S + \tilde{g}DF,$$
$$\dot{D} = -2\tilde{g}(SF^* + S^*F) - 2\gamma(D-1), \quad (3)$$

with $\kappa_{\text{eff}} = \kappa(1+\tilde{C}')$, which are the so-called Maxwell-Bloch equations well known in laser physics [29,30]. The steady-state solution of these equations predicts a lasing transition depending on the cooperativity parameter $\tilde{C} = \tilde{g}^2/\gamma\kappa_{\text{eff}}$, which separates a trivial phase with $\bar{F} = \bar{S} = 0$ and $\bar{D} = 1$ (bar indicates steady-state values within the mean-field approximation) for $\tilde{C} < 1$, from a bright phase when $\tilde{C} > 1$ in which $\bar{F} = \sqrt{\gamma(\tilde{C}-1)/2\kappa_{\text{eff}}\tilde{C}}\exp(i\theta)$, $\bar{S} = \tilde{g}\bar{F}/\tilde{C}\gamma$, and $\bar{D} = 1/\tilde{C}$, where θ is an arbitrary phase not fixed by the equations. Note that deep into the lasing regime ($\tilde{C} \to \infty$) the number of (mean-field) photons depends solely on the ratio $\kappa_{\text{eff}}/\gamma$.

The mean-field approximation also allows us to estimate the reduced steady state of the field $\rho_{\text{f}} = \text{tr}_{\text{qubit}}\{\rho\}$. For this, we just use the fact that within this approximation the state is separable in the qubit-field subspaces, so that taking the partial trace of (2), we get a Gaussian master equation with an analytic stationary solution; in particular, using the parametrization $\{u = \cosh r, v = \sinh r\}$ with $r \in [0, \infty]$ (we assume from now on that $u, v > 0$ without a loss of generality), and taking into account that the mean-field solution assumes spontaneous symmetry breaking, whereas in reality the statistics over many realizations would show a random phase θ, we show in the Supplemental Material [27,33] that our mean-field ansatz is given by the mixture

$$\bar{\rho}_{\text{f}}(|\bar{F}|, \tilde{C}', r) = \int_0^{2\pi}\frac{d\theta}{2\pi}D_A(\bar{F})S_A(\tilde{r})\rho_{\text{th},A}(\tilde{n})S_A^\dagger(\tilde{r})D_A^\dagger(\bar{F}), \quad (4)$$

where we have defined the displacement and squeezing operators $D_A(\alpha) = \exp(\alpha A^\dagger - \alpha^* A)$ and $S_A(r) = \exp[r(A^{\dagger 2} - A^2)/2]$, respectively, and the thermal state $\rho_{\text{th},A}(N) = \sum_{n=0}^\infty [N^n/(1+N)^{1+n}]|n\rangle_A\langle n|$, $|n\rangle_A$ referring to the Fock states associated to mode A, and where $4\tilde{r} = \ln\{[\exp(2r) + \tilde{C}']/[\exp(-2r) + \tilde{C}']\}$ and $2\tilde{n} = \sqrt{[\exp(2r) + \tilde{C}'][\exp(-2r) + \tilde{C}']/(1+\tilde{C}')^2} - 1$.

This mean-field state is a generalization of the usual coherent-state mixture found in the laser [29,31,32]; below, we discuss how well it describes the system compared to the exact steady state, but, before doing so, let us consider two physically relevant limits. First, the limit $\tilde{C}' \gg \exp(2r)$, in which $\tilde{r} = 0$ and $\tilde{n} = 0$, so that the ansatz can be written as

$$\bar{\rho}_{\text{f}}^{(1)} = \int_0^{2\pi}\frac{d\theta}{2\pi}D_A(\bar{F})|0\rangle_A\langle 0|D_A^\dagger(\bar{F})$$
$$= \int_0^{2\pi}\frac{d\theta}{2\pi}S_a^\dagger(r)D_a(\bar{F})|0\rangle_a\langle 0|D_a^\dagger(\bar{F})S_a(r); \quad (5)$$

we see that, as expected, in this limit the state is just a balanced mixture of all the coherent states of mode A with the same mean-field amplitude $|\bar{F}|$, which is the ideal laser state. Hence, this is the limit in which our system works as a nonclassical laser, since this state corresponds to a mixture of squeezed coherent states in the basis of the original cavity mode. The second limit we want to consider is $\tilde{C}' \to 0$, that is, the limit in which we do not add a second qubit to engineer dissipation in the squeezed mode A. In this case, $\tilde{n} = 0$ again, but $\tilde{r} = r$, so that the mean-field ansatz can be written as

$$\bar{\rho}_{\text{f}}^{(2)} = \int_0^{2\pi}\frac{d\theta}{2\pi}D_A(\bar{F})S_A(r)|0\rangle_A\langle 0|S_A^\dagger(r)D_A^\dagger(\bar{F})$$
$$= \int_0^{2\pi}\frac{d\theta}{2\pi}D_a(u\bar{F} - v\bar{F}^*)|0\rangle_a\langle 0|D_a^\dagger(u\bar{F} - v\bar{F}^*); \quad (6)$$

this shows that without the help of the second qubit, the lasing process is still classical from the point of view of the original mode a, that is, the state is a mixture of coherent states.

Our laser works in a mesoscopic photon number regime in which the validity of the mean-field solution must be handled with care, and hence we proceed to study numerically the exact steady state of (2). In the solid blue curve of Fig. 2(a), we show the fidelity between this exact steady state and the mean-field ansatz (4) as we move up into the lasing transition for 90% of quadrature squeezing ($r \approx 1.15$), $\kappa_{\text{eff}}/\gamma = 0.02$, and $\tilde{C}' = 10$ (similar curves are found for other values of $\tilde{C}'$). It can be appreciated how the mean-field ansatz adapts very well to the exact steady state above the lasing transition. In addition, in the rest of the curves of Fig. 2(a), we show the fidelity between the ansatz and the exact steady state of the system when the second qubit is not adiabatically eliminated, for different

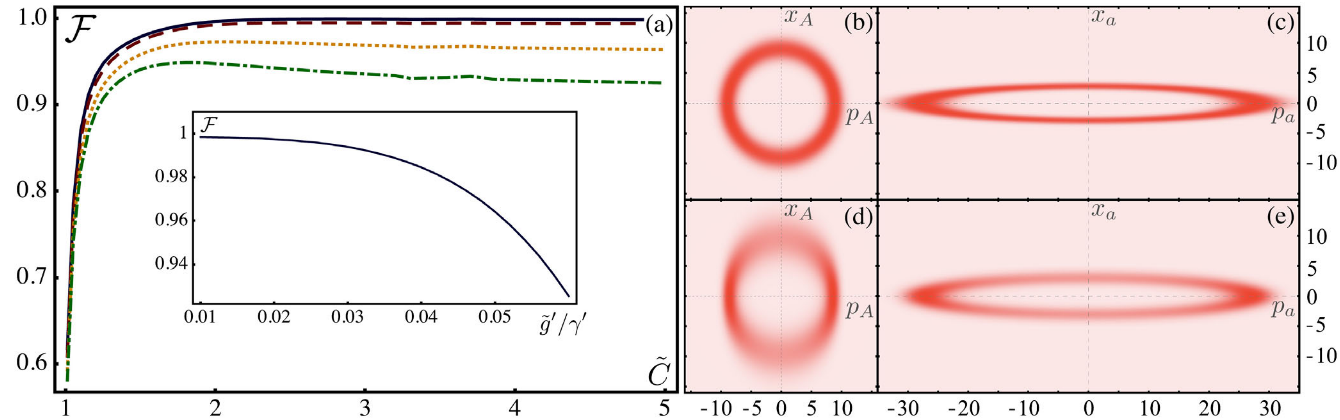

FIG. 2 (color online). (a) Fidelity between the mean-field ansatz (4) and the exact steady state of the system, as a function of the cooperativity $\tilde{C}$, for $\kappa_{\text{eff}}/\gamma = 0.02$, $r \approx 1.15$ (90% of quadrature squeezing), and $\tilde{C}' = 10$. The solid blue curve corresponds to the exact steady state of Eq. (2), while in the other curves the effect of the second qubit is considered for $\tilde{g}'/\gamma' = 0.02$ (dashed red line), 0.05 (dotted yellow line), and 0.07 (dash-dotted green line); the inset shows the fidelity as a function of the ratio $\tilde{g}'/\gamma'$, fixing the parameters as in the main plot, plus $\tilde{C} = 5$. We also show density plots of the Wigner functions corresponding to the steady state of (2) for $\tilde{C} = 5$ and $\kappa_{\text{eff}}/\gamma = 0.02$, and two values of $\tilde{C}'$, 10 (b),(c) and 0.01 (d),(e), in which the states are well approximated by (5) and (6), respectively. Note that since (b) approximately corresponds to a mixture of coherent states of equal amplitude, its width can be used as the unit for shot noise, thus making apparent the squeezing in (c).

values of $\tilde{g}'/\gamma'$; we can appreciate that $\tilde{g}'/\gamma' \lesssim 0.03$ is needed in order to achieve the lasing conditions we seek.

In order to better characterize the state in the different regimes, Figs. 2(b)–2(e) show the Wigner functions corresponding to the limiting situations $\tilde{C}' \ll 1$ and $\tilde{C}' \gg v$, characterized by states (5) and (6), respectively—see [27,33–36] for the details of their evaluation. In particular, in Figs. 2(d) and 2(e) we plot these Wigner functions in the phase space of the original cavity mode a, which we propose to reconstruct with a tomography experiment along the lines of [6,37]. Since all of our Wigner functions are positive everywhere in the phase space formed by the quadratures $x_c = c^\dagger + c$ and $p_c = i(c^\dagger - c)$, where $c = A$ or a, they can be directly interpreted as the joint probability distribution describing measurements of these observables.

Let us finally remark that it can be proved that the addition of dephasing (at rate γ_z) in the qubits doesn't have any significant effect on the results presented above, except for decreasing the cooperativity of the corresponding qubit by a factor $(1 + 4\gamma_z/\gamma)$. Consequently, we have omitted such processes to ease the presentation.

Physical implementation.—In order to show the feasibility of our scheme, let us give some concrete parameters for the circuit QED architecture sketched in Fig. 1. We take $\varepsilon/2\pi = 10$ GHz, $\omega/2\pi = 4.5$ GHz, and $g/2\pi = 40$ MHz, which are common parameters in state-of-the-art superconducting circuits [37]. In addition, a relatively fast radiative decay rate $\gamma/2\pi = 15$ MHz is induced in the qubit, while the LC resonator has a damping rate $\kappa/2\pi = 30$ kHz to the readout circuit. As for the driving, let us fix $\eta_2 = 0.2$, corresponding to a modulation amplitude of $\Omega_2\eta_2 = 2.9$ GHz, which is quite reasonable. Single-qubit lasing with cooperativities and photon numbers up to 140 and 250, respectively, can be achieved with these parameters. In order to generate the squeezed lasing proposed in the Letter, one could include a second qubit with $g'/2\pi = 70$ MHz and a strong radiative decay $\gamma'/2\pi = 250$ MHz, conditions in which its adiabatic elimination should be valid. Having chosen a small normalized amplitude η_2, we can approximate $\tanh r = v/u \approx \eta_1/\eta_2$, so that 90% of quadrature squeezing ($r \approx 1.15$) is obtained by choosing $\eta_1 \approx 0.16$; taking into account that the renormalized coupling can be approximated by $\tilde{g} \approx g\sqrt{\eta_2^2 - \eta_1^2}$ (and similarly for $\tilde{g}'$), one can then get up to cooperativities $\tilde{C} = 5$ and $\tilde{C}' = 10$, enough to see the phenomena introduced in the Letter.

Conclusions and Outlook.—We have shown how to engineer a single-atom laser that emits light into a nonclassical state in a circuit QED scenario. Our scheme relies only on the modulation of the transition frequencies of two qubits with periodic drivings and exploits their radiative decay to our advantage: for one qubit it is turned into the effective population inversion mechanism needed for lasing, while for the other it allows engineering the cavity dissipation such that the lasing process becomes nonclassical. The generalization of our ideas to the generation of multimode squeezed and entangled states is straightforward, while the extension of our work to many qubits would allow studying strongly correlated phenomena with circuit QED setups, providing the exciting possibility of preparing nontrivial many-body states dissipatively [38–42]. In addition to this, working in the GHz range, where stable phase-locked continuous-wave sources exist, our system opens a way to analyzing the coherence of the lasing process and its underlying spontaneous symmetry breaking mechanism, questions upon which there are still open debates among the quantum optics community [43–45].

We thank Géza Giedke for the interesting suggestions concerning the characterization of the state of the system; Irfan Siddiqi, Steven J. Weber, and Kater W. Murch, for insight about the concrete circuit QED architecture and the corresponding system parameters; and Alfredo Luis for the fruitful discussions. We acknowledge CAM Research Consortium QUITEMAD (Grant No. S2009- ESP-1594). C. N.-B. acknowledges the support from the Alexander von Humboldt Foundation through its Fellowship for Postdoctoral Researchers, and of the Future and Emerging Technologies (FET) program within the Seventh Framework Programme for Research of the European Commission under the FET- Open grant agreement MALICIA, Grant No. FP7-ICT- 265522. J. J. G.-R acknowledges FET-Open project PROMISCE and MINECO Project No. FIS2012-33022. D. P. acknowledges RyC Contract No. Y200200074.

[1] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).

[2] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature (London) **431**, 162 (2004).

[3] O. Astafiev, K. Inomata, A. O. Niskanen, T. Yamamoto, Y. A. Pashkin, Y. Nakamura, and J. S. Tsai, Nature (London) **449**, 588 (2007).

[4] M. Grajcar, S. H. W. van der Ploeg, A. Izmalkov, E. Il'ichev, H.-G. Meyer, A. Fedorov, A. Shnirman, and G. Schön, Nat. Phys. **4**, 612 (2008).

[5] M. A. Castellanos-Beltran, K. D. Irwin, G. C. Hilton, L. R. Vale, and K. W. Lehnert, Nat. Phys. **4**, 929 (2008).

[6] F. Mallet, M. A. Castellanos-Beltran, H. S. Ku, S. Glancy, E. Knill, K. D. Irwin, G. C. Hilton, L. R. Vale, and K. W. Lehnert, Phys. Rev. Lett. **106**, 220502 (2011).

[7] K. W. Murch, S. J. Weber, K. M. Beck, E. Ginossar, and I. Siddiqi, Nature (London) **499**, 62 (2013).

[8] D. Porras and J. J. García-Ripoll, Phys. Rev. Lett. **108**, 043602 (2012).

[9] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Rev. Mod. Phys. **82**, 1041 (2010).

[10] J. Li, M. Silveri, K. Kumar, J.-M. Pirkkalainen, A. Vepsalainen, W. Chien, J. Tuorila, M. Sillanpaa, P. Hakonen, E. Thuneberg, and G. Paraoanu, Nat. Commun. **4**, 1420 (2013).

[11] P. Lähteenmäki, G. S. Paraoanu, J. Hassel, and P. J. Hakonen, Proc. Natl. Acad. Sci. U.S.A. **110**, 4234 (2013).

[12] D. Ballester, G. Romero, J. J. García-Ripoll, F. Deppe, and E. Solano, Phys. Rev. X **2**, 021007 (2012).

[13] K. W. Murch, U. Vool, D. Zhou, S. J. Weber, S. M. Girvin, and I. Siddiqi, Phys. Rev. Lett. **109**, 183602 (2012).

[14] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[15] V. V. Dodonov, J. Opt. B **4**, R1 (2002).

[16] J. I. Cirac, A. S. Parkins, R. Blatt, and P. Zoller, Phys. Rev. Lett. **70**, 556 (1993).

[17] J. McKeever, A. Boca, A. D. Boozer, J. R. Buck, and H. J. Kimble, Nature (London) **425**, 268 (2003).

[18] M. I. Dykman, Sov. J. Low Temp. Phys. **5**, 186 (1979).

[19] T. M. Stace, A. C. Doherty, and S. D. Barrett, Phys. Rev. Lett. **95**, 106801 (2005).

[20] Z. G. Xie, S. Götzinger, W. Fang, H. Cao, and G. S. Solomon, Phys. Rev. Lett. **98**, 117401 (2007).

[21] M. Nomura, N. Kumagai, S. Iwamoto, Y. Ota, and Y. Arakawa, Nat. Phys. **6**, 279 (2010).

[22] J. I. Colless, X. G. Croot, T. M. Stace, A. C. Doherty, S. D. Barrett, H. Lu, A. C. Gossard, and D. J. Reilly, arXiv: 1305.5982.

[23] S. Schmidt and J. Koch, Ann. Phys. (Amsterdam) **525**, 395 (2013).

[24] F. Quijandría, D. Porras, J. J. García-Ripoll, and D. Zueco, Phys. Rev. Lett. **111**, 073602 (2013).

[25] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **76**, 042319 (2007).

[26] J. A. Schreier, A. A. Houck, J. Koch, D. I. Schuster, B. R. Johnson, J. M. Chow, J. M. Gambetta, J. Majer, L. Frunzio, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. B **77**, 180502 (2008).

[27] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.113.193601 for a derivation of the effective time-independent Hamiltonian and of the mean-field steady state, as well as the relation between the density matrix in the Fock basis and the Wigner function.

[28] M. Grifoni and P. Hänggi, Phys. Rep. **304**, 229 (1998).

[29] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).

[30] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, New York, 2002).

[31] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).

[32] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, New York, 1994).

[33] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[34] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).

[35] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).

[36] B. M. Garraway and P. L. Knight, Phys. Rev. A **46**, R5346 (1992).

[37] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O'Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature (London) **459**, 546 (2009).

[38] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Buchler, and P. Zoller, Nat. Phys. **4**, 878 (2008).

[39] F. Verstraete, M. M. Wolf, and J. I. Cirac, Nat. Phys. **5**, 633 (2009).

[40] J. T. Barreiro, P. Schindler, O. Guhne, T. Monz, M. Chwalla, C. F. Roos, M. Hennrich, and R. Blatt, Nat. Phys. **6**, 943 (2010).

[41] J. T. Barreiro, M. Muller, P. Schindler, D. Nigg, T. Monz, M. Chwalla, M. Hennrich, C. F. Roos, P. Zoller, and R. Blatt, Nature (London) **470**, 486 (2011).

[42] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Phys. Rev. Lett. **107**, 080503 (2011).

[43] K. Mølmer, Phys. Rev. A **55**, 3195 (1997).

[44] T. Rudolph and B. C. Sanders, Phys. Rev. Lett. **87**, 077903 (2001).

[45] S. J. van Enk and C. A. Fuchs, Phys. Rev. Lett. **88**, 027902 (2001).

Supplemental material: Inducing Non-Classical Lasing Via Periodic Drivings in Circuit Quantum Electrodynamics

Carlos Navarrete-Benlloch,[1] Juan José García-Ripoll,[2] and Diego Porras[3,4]

[1] *Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-strasse 1, 85748 Garching, Germany*
[2] *Instituto de Física Fundamental, IFF-CSIC, Serrano 113-bis, Madrid E-28006, Spain*
[3] *Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN19QH, UK*
[4] *Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain*

(Dated: October 1, 2014)

In this supplemental material we offer a detailed derivation of three points of the main Letter: (i) the time-independent Hamiltonian which captures the dynamics induced by the full time-dependent Hamiltonian modeling the driven qubit-field system; (ii) the solution of the field's steady-state within the mean-field approximation; and (iii) the construction of the Wigner functions from the density matrices obtained numerically in the Fock basis.

EFFECTIVE TIME-INDEPENDENT HAMILTONIAN

In the main Letter, we claimed that the dynamics induced by the time-dependent Hamiltonian $H(t) = H_0 + H_{\text{int}} + H_{\text{d}}(t)$, with

$$H_0 = \omega a^\dagger a + \frac{\varepsilon}{2}\sigma_z, \quad H_{\text{int}} = g(a + a^\dagger)\sigma_x,$$
$$H_{\text{d}}(t) = \sum_{j=1}^{2} \Omega_j \eta_j \cos(\Omega_j t)\sigma_z, \tag{1}$$

is well captured by the time-independent one

$$\tilde{H} = -\tilde{g}\left(ua^\dagger + va\right)\sigma^\dagger + \text{H.c.}, \tag{2}$$

where $\tilde{g}$ is a renormalized coupling and the parameters u and v satisfy the Bogoliubov relation $|u^2 - v^2| = 1$, provided that one works far from the strong-coupling regime and off-resonance ($\omega, \varepsilon, |\varepsilon - \omega| \gg g$), and chooses the upper and lower sideband modulations $\Omega_{1,2} = \varepsilon \mp \omega$. In this first section of the supplemental material we prove this statement rigorously.

To this aim, let us first move to the interaction picture defined by the transformation operator

$$\begin{aligned} U_{\text{c}}(t) &= \exp\left[-iH_0 t - i\int_0^t d\tau H_{\text{d}}(\tau)\right] \\ &= \exp\left[-i\omega t a^\dagger a - i\left(\frac{\varepsilon t}{2} + \sum_{j=1}^{2}\eta_j \sin\Omega_j t\right)\sigma_z\right], \end{aligned} \tag{3}$$

which transforms the state of the qubit-field system as $\rho \to \rho_{\text{I}} = U_{\text{c}}^\dagger \rho U_{\text{c}}$, so that it evolves now according to the Hamiltonian

$$\begin{aligned} H_{\text{I}} &= U_{\text{c}}^\dagger[H_0 + H_{\text{d}}(t)]U_{\text{c}} - H_0 - H_{\text{d}}(t) \\ &= g\left\{a\sigma \exp\left[-i\left(\omega t + \varepsilon t + \sum_{j=1}^{2} 2\eta_j \sin\Omega_j t\right)\right]\right. \\ &\quad \left. + a\sigma^\dagger \exp\left[-i\left(\omega t - \varepsilon t - \sum_{j=1}^{2} 2\eta_j \sin\Omega_j t\right)\right]\right\} + \text{H.c.}, \end{aligned} \tag{4}$$

where we have used

$$U_{\text{c}}^\dagger a U_{\text{c}} = a\exp(-i\omega t), \tag{5}$$
$$U_{\text{c}}^\dagger \sigma U_{\text{c}} = \sigma\exp\left[-i\left(\varepsilon t + \sum_{j=1}^{2} 2\eta_j \sin\Omega_j t\right)\right]. \tag{6}$$

The next step in the derivation consists in using the fact that the sine function is the generator of the Bessel functions, what means that

$$\exp(2i\eta_j \sin\Omega_j t) = \sum_{n=-\infty}^{+\infty} J_n(2\eta_j)\exp(in\Omega_j t), \tag{7}$$

leading to the Hamiltonian

$$H_{\text{I}} = \hbar g\left[\alpha(t)a\sigma^\dagger + \beta(t)a\sigma\right] + \text{H.c.}, \tag{8}$$

with

$$\alpha(t) = \sum_{n_1,n_2=-\infty}^{+\infty} J_{n_1}(2\eta_1)J_{n_2}(2\eta_2)e^{-i(\omega-\varepsilon-n_1\Omega_1-n_2\Omega_2)t}, \tag{9a}$$

$$\beta(t) = \sum_{n_1,n_2=-\infty}^{+\infty} J_{n_1}(2\eta_1)J_{n_2}(2\eta_2)e^{-i(\omega+\varepsilon+n_1\Omega_1+n_2\Omega_2)t}. \tag{9b}$$

This Hamiltonian has both rotating ($a\sigma^\dagger$) and counter-rotating ($a\sigma$) terms; however, these terms will contribute to the dynamics of the system only if some of the complex exponentials appearing in the definition of $\alpha(t)$ and $\beta(t)$ vary slowly compared to g (rotating-wave approximation), that is, introducing $\Omega_{1,2} = \varepsilon \mp \omega$, the rotating term will contribute only for (m_1, m_2) such that

$$|(1 + m_1 + m_2)\omega - (1 + m_1 - m_2)\varepsilon| \ll g, \tag{10}$$

while the counter-rotating term will enter the dynamics only if

$$|(1-q_1+q_2)\omega+(1+q_1+q_2)\varepsilon|\ll g, \tag{11}$$

for some combination (q_1,q_2). It is possible to find exponentials which oscillate slow compared to g both in $\alpha(t)$ and $\beta(t)$. In particular, provided the no multi-photon resonances are allowed within the coupling strength, that is

$$|m\varepsilon-n\omega|\gg g \quad \forall mn=1,2,..., \tag{12}$$

only one term of $\alpha(t)$ and another of $\beta(t)$ survive, the ones with $(m_1=-1,m_2=0)$ and $(q_1=0,q_2=-1)$, respectively. Note however that it is enough that condition (12) holds for small m and n, as if the multi-photon resonance occurs only for large ones, only high order Bessel functions kick in, and then the terms previously found are still the only ones which contribute to $\alpha(t)$ and $\beta(t)$ approximately. For example, for the frequencies chosen in the Letter, $\varepsilon/2\pi=10$GHz and $\omega/2\pi=4.5$GHz, the first multi-photon resonance that satisfies (10) is $(m_1=28,m_2=11)$, which gives a negligible contribution to $\alpha(t)$ unless the modulation amplitudes η_j are extremely large.

Under such conditions, the Hamiltonian (8) takes the form

$$H_{\rm I}\approx g\left[J_{-1}(2\eta_1)J_0(2\eta_2)a+J_0(2\eta_1)J_{-1}(2\eta_2)a^\dagger\right]\sigma^\dagger+\text{H.c.}; \tag{13}$$

now, using the property $J_{-1}(x)=-J_1(x)$, and defining the parameters

$$v = \frac{J_1(2\eta_1)J_0(2\eta_2)}{\sqrt{|J_1^2(2\eta_1)J_0^2(2\eta_2)-J_0^2(2\eta_1)J_1^2(2\eta_2)|}}, \tag{14a}$$

$$u = \frac{J_0(2\eta_1)J_1(2\eta_2)}{\sqrt{|J_1^2(2\eta_1)J_0^2(2\eta_2)-J_0^2(2\eta_1)J_1^2(2\eta_2)|}}, \tag{14b}$$

$$\tilde{g} = g\sqrt{|J_1^2(2\eta_1)J_0^2(2\eta_2)-J_0^2(2\eta_1)J_1^2(2\eta_2)|}, \tag{14c}$$

we obtain the Hamiltonian (2) as we wanted to prove.

GAUSSIAN-STATE SOLUTION TO THE MEAN-FIELD EQUATION

In this section we find the stationary solution for the state of the field within the mean-field approximation. As explained in the Letter, the master equation of the system (under adiabatic elimination of the second qubit) is

$$\dot\rho=i\tilde{g}[A^\dagger\sigma^\dagger+A\sigma,\rho]+\mathcal{L}_{\{\sigma,\gamma\}}[\rho]+\mathcal{L}_{\{a,\kappa\}}[\rho]+\mathcal{L}_{\{A,\kappa\tilde{C}'\}}[\rho]. \tag{15}$$

The mean-field approximation consists in assuming that the state is separable in the qubit-field subspaces, and we show in the main Letter that, defining the expectation values $F=\langle A\rangle$ and $S=i\langle\sigma\rangle^*$, these approximation predicts a lasing transition depending on the cooperativity parameter $\tilde{C}=\tilde{g}^2/\gamma\kappa(1+\tilde{C}')$, which separates a trivial phase with $\bar{F}=\bar{S}=0$ (from now on the bar indicates steady-state values within the mean-field approximation) for $\tilde{C}<1$, from a bright phase when $\tilde{C}>1$ in which

$$\bar{F}=\sqrt{\frac{\gamma(\tilde{C}-1)}{2\kappa(1+\tilde{C}')\tilde{C}}}e^{i\theta},\quad \bar{S}=\frac{\tilde{g}}{\tilde{C}\gamma}\bar{F}, \tag{16}$$

where θ is an arbitrary phase not fixed by the equations. In this case the mean-field approximation even allows us to estimate the reduced steady state of the field $\rho_{\rm f}=\text{tr}_{\rm qubit}\{\rho\}$. For this, we just use the fact that within this approximation the state is separable in the qubit-field subspaces, so that taking the partial trace of (15), and using (16), we get

$$\dot\rho_{\rm f}=(1+\tilde{C}')[\bar{F}A^\dagger-\bar{F}^*A,\rho_{\rm f}]+\mathcal{L}_{\{a,1\}}[\rho_{\rm f}]+\mathcal{L}_{\{A,\tilde{C}'\}}[\rho_{\rm f}], \tag{17}$$

with $a=A\cosh r-A^\dagger\sinh r$. Note that given any field operator O, we can find the evolution equation of its expectation value as

$$\begin{aligned}\langle\dot{O}\rangle &= \text{tr}\{O\dot\rho_{\rm f}\}=(1+C')\langle[O,\bar{F}A^\dagger-\bar{F}^*A]\rangle+\langle a^\dagger[O,a]\rangle\\ &\quad+\langle[a^\dagger,O]a\rangle+\tilde{C}'\langle A^\dagger[O,A]\rangle+\tilde{C}'\langle[A^\dagger,O]A\rangle. \end{aligned}\tag{18}$$

Now, since equation (17) is quadratic in annihilation and creation operators $(A,A^\dagger)$, its steady state $\bar\rho_{\rm f}$ is Gaussian, meaning that it is completely characterized by its first and second moments [1]. In particular, using (18) it is simple to find $\overline{\langle A\rangle}=\bar{F}$,

$$\overline{\langle A^\dagger A\rangle} = |\bar{F}|^2+\frac{\sinh^2 r}{1+\tilde{C}'}, \tag{19a}$$

$$\overline{\langle A^2\rangle} = \bar{F}^2+\frac{\sinh 2r}{2(1+\tilde{C}')}. \tag{19b}$$

Defining the quadratures $x_A=A^\dagger+A$ and $p_A=i(A^\dagger-A)$, the vector operator $\mathbf{r}_A=\text{col}(x_A,p_A)$, and the corresponding mean vector $\mathbf{d}_A=\langle\mathbf{r}_A\rangle$ and covariance matrix V_A with elements $V_{A,jk}=\langle r_{A,j}r_{A,k}\rangle-\langle r_{A,j}\rangle\langle r_{A,k}\rangle$, we then get a state with Gaussian Wigner function

$$\bar{W}_{\rm f}(\mathbf{R}_A)=\frac{1}{2\pi\sqrt{\det\bar{V}_A}}e^{-(\mathbf{R}_A-\bar{\mathbf{d}}_A)^T\bar{V}_A^{-1}(\mathbf{R}_A-\bar{\mathbf{d}}_A)/2}, \tag{20}$$

where $\mathbf{R}_A=\text{col}(X_A,P_A)$ are phase space variables associated to the quadrature operators (in the main Letter we kept the names x_A and p_A for these c-numbers in Fig. 2 for simplicity), and

$$\bar{\mathbf{d}}_A = 2\text{col}(\text{Re}\{\bar{F}\},\text{Im}\{\bar{F}\}), \tag{21a}$$

$$\bar{V}_A = \frac{1}{\tilde{C}'+1}\begin{pmatrix}\tilde{C}'+e^{2r} & 0\\ 0 & \tilde{C}'+e^{-2r}\end{pmatrix}. \tag{21b}$$

In order to gain more insight, we are going to write this Gaussian state in a different manner. Concretely, it is well known that any single-mode Gaussian state can always be written in the form [1]

$$\rho = D_A(\alpha)R_A(\varphi)S_A(\tilde{r})\rho_{\text{th},A}(\tilde{n})S_A^\dagger(\tilde{r})R_A^\dagger(\varphi)D_A^\dagger(\alpha), \tag{22}$$

where we have defined the displacement $D_A(\alpha) = \exp(\alpha A^\dagger - \alpha^* A)$, phase-shift $R_A(\varphi) = \exp(i\varphi A^\dagger A)$, and squeezing $S_A(\tilde{r}) = \exp[\tilde{r}(A^{\dagger 2} - A^2)/2]$ operators, as well as the thermal state $\rho_{\text{th},A}(\tilde{n})$, which is a Gaussian state with zero mean vector and covariance matrix $V_{\text{th},A}(\tilde{n}) = (2\tilde{n}+1)I_{2\times 2}$. The displacement parameter α coincides with the mean of the state, what in our case means $\alpha = \bar{F}$, while no phase-shift is needed ($\varphi = 0$) for a state with a diagonal covariance matrix as (21b). On the other hand, since the entropy is invariant under unitary transformations, and for a single-mode state it depends solely on the determinant of the covariance matrix [1], the thermal photon number parameter $\tilde{n}$ is found by matching the determinants of $V_{\text{th},A}(\tilde{n})$ and $\bar{V}_A$, that is, $(2\tilde{n}+1)^2 = \det \bar{V}_A$. Finally, $S_A(\tilde{r})$ squeezes (anti-squeezes) the momentum (position) variance by a factor $e^{-2\tilde{r}}$ ($e^{2\tilde{r}}$), and hence the squeezing parameter $\tilde{r}$ is found from the asymmetry of the covariance matrix, that is, $\exp(4\tilde{r}) = \bar{V}_{A,11}/\bar{V}_{A,22}$. Combining all these results, (22) is turned into the Gaussian state

$$\rho_{\text{f}}^{\text{G}}(\bar{F}, \tilde{C}', r) = D_A(\bar{F})S_A(\tilde{r})\rho_{\text{th},A}(\tilde{n})S_A^\dagger(\tilde{r})D_A^\dagger(\bar{F}). \tag{23}$$

Then, taking into account that the mean-field solution (16) assumes spontaneous symmetry breaking, whereas in reality the statistics over many realizations would show a random phase θ, the mean-field ansatz has to be taken as the mixture

$$\bar{\rho}_{\text{f}}(|\bar{F}|, \tilde{C}', r) = \int_0^{2\pi} \frac{d\theta}{2\pi} \rho_{\text{f}}^{\text{G}}(\bar{F}, \tilde{C}', r), \tag{24}$$

precisely the state that we introduced in the Letter.

Note finally that, given the relation $A = S_a^\dagger(r) a S_a(r)$, the relation between the Fock basis of the squeezed and original cavity modes is $|n\rangle_A = S_a^\dagger(r)|n\rangle_a$, and hence the Gaussian state (23) can be written as

$$\rho_{\text{f}}^{\text{G}} = S_a^\dagger(r)D_a(\alpha)S_a(\tilde{r})\rho_{\text{th},a}(\tilde{n})S_a^\dagger(\tilde{r})D_a^\dagger(\alpha)S_a(r), \tag{25}$$

in the basis of the original cavity mode.

WIGNER FUNCTIONS FROM THE DENSITY MATRIX

All our numerics have been performed by using the Fock states $\{|n\rangle_A\}_{n=0,1,\dots,N_A}$ of the squeezed mode A as the basis of the field's Hilbert space (truncated to a large enough photon number N_A), what gives us the reduced state of the cavity mode represented as $\rho_{\text{f}} = \sum_{mn=0}^{N_A} \rho_{mn}^A |m\rangle_A\langle n|$. In this section we explain how to find the Wigner functions in the phase space of both the squeezed mode A and the original cavity mode a, starting from this representation of the state.

Let us write the polar form of the coordinate vector in the phase space of mode A as $\mathbf{R}_A = R_A(\cos\phi_A, \sin\phi_A)$. Hence, based on the following result [2–4] for the Wigner function of the operator $|m\rangle_A\langle n|$:

$$W_{mn}(R_A, \phi_A) = \frac{(-1)^n}{\pi} \sqrt{\frac{n!}{m!}} e^{i\phi_A(m-n)} R_A^{m-n} \times L_n^{m-n}(R_A^2) e^{-R_A^2/2}, \tag{26}$$

where $L_n^p(x)$ are the modified Laguerre polynomials and we have assumed $m \geq n$ (note that $W_{nm} = W_{mn}^*$), we get

$$W_A(R_A, \phi_A) = \sum_{mn=0}^{N_A} \rho_{mn}^A W_{mn}(R_A, \phi_A), \tag{27}$$

which gives us the desired relation between the density matrix ρ^A and the Wigner function $W_A(\mathbf{R}_A)$ in the phase space of the squeezed mode A.

On the other hand, in order to find the Wigner function in the phase space of the original cavity mode a, we just use the fact that $A = S_a^\dagger(r) a S_a(r)$ is equivalent to the symplectic transformation [1] $\mathbf{R}_A = \mathcal{S}(r)\mathbf{R}_a$ between the corresponding phase spaces, with $\mathcal{S}(r) = \text{diag}(e^r, e^{-r})$. Hence, given the Wigner function evaluated with (27) in the phase space of mode A, the Wigner function in the phase space of mode a is found as $W_a(\mathbf{R}_a) = W_A[\mathcal{S}(r)\mathbf{R}_a]$.

[1] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[2] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).

[3] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).

[4] B. M. Garraway and P. L. Knight, Phys. Rev. A **46**, R5346 (1992).